More Natural Derivations for Priest, *An Introduction to Non-Classical Logic, 2nd ed.*

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In [7] Roy produced natural derivation systems, including demonstration of soundness and completeness, for each of the logics described in the first edition of Priest, *An Introduction to Non-Classical Logic* [3]. The first edition of Priest's book is Part I of the second edition. Eventually, we hope to complete the project, providing natural derivation systems for the quantified versions in Part II. In the meantime, this document simply extends the previous paper to account for additions and changes in the first part of the new edition.

Thus, as before, we offer an alternative or supplement to the semantic tableaux of his text. Some of the derivation systems may also be of interest in their own right. They are all Fitch-style systems on the model of [1, 6], and many other places. Though a classical system is presented for chapter 1, prior acquaintance with some such system is assumed. Associated goal-directed derivation strategies are discussed extensively in [6, chapter 6].

Except that some chapters are collapsed, there are sections for each chapter in the first part of Priest's book, with an additional section on four-valued relevant logic. In each case, (i) the language is briefly described and key semantic definitions stated, (ii) the derivation system is presented with a few examples given, and (iii) soundness and completeness are proved.

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*Thanks to all!

Contents

1 Classical Logic: *CL* (ch. 1)

1.1 Language / Semantic Notions

LCL The LANGUAGE consists of propositional parameters $p_0, p_1 \ldots$ combined in the usual way with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, and $\equiv$. So each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$ and $(A \equiv B)$.

ICL An INTERPRETATION is a function v which assigns to each propositional parameter either 1 (true) or 0 (false).

TCL For complex expressions,

($\neg$) $v(\neg A) = 1$ if $v(A) = 0$, and 0 otherwise.
($\wedge$) $v(A \wedge B) = 1$ if $v(A) = 1$ and $v(B) = 1$, and 0 otherwise.
($\vee$) $v(A \vee B) = 1$ if $v(A) = 1$ or $v(B) = 1$, and 0 otherwise.
($\supset$) $v(A \supset B) = 1$ if $v(A) = 0$ or $v(B) = 1$, and 0 otherwise.
($\equiv$) $v(A \equiv B) = 1$ if $v(A) = v(B)$, and 0 otherwise.

For a set Γ of formulas, $v(\Gamma) = 1$ iff $v(A) = 1$ for each $A \in \Gamma$; then,

VCL $\Gamma \models_{CL} A$ iff there is no *CL* interpretation v such that $v(\Gamma) = 1$ and $v(A) = 0$.

1.2 Natural Derivations: *NCL*

NCL is just the sentential portion of the system *ND* from [6, chapter 6]. Refer to that source for examples and further discussion (compare, e.g., [1]). Every line of a derivation is a premise, an assumption, or justified from previous lines by a rule. The rules include *introduction* and *exploitation* rules for each operator, and *reiteration*. In the parenthetical "exit strategy" for assumptions, '*c*' indicates a contradiction is to be sought, '*g*' a goal at the bottom of the scope line.

R (*reiteration*)

```
a | P
  |
  |
  | P          a R
```

¬I (*negation intro*)

```
a | | P        A (c, ¬I)
  | |-
  | |
  | | Q
b | | ¬Q
  | ¬P         a-b ¬I
```

¬E (*negation exploit*)

```
a | | ¬P       A (c, ¬E)
  | |-
  | |
  | | Q
b | | ¬Q
  | P          a-b ¬E
```

$\wedge$**I** (*conjunction intro*)

$$\begin{array}{r|ll} a & P & \\ b & Q & \\ & & \\ & P \wedge Q & a,b\ \wedge\text{I} \end{array}$$

$\wedge$**E** (*conjunction exploit*)

$$\begin{array}{r|ll} a & P \wedge Q & \\ & & \\ & P & a\ \wedge\text{E} \end{array}$$

$\wedge$**E** (*conjunction exploit*)

$$\begin{array}{r|ll} a & P \wedge Q & \\ & & \\ & Q & a\ \wedge\text{E} \end{array}$$

$\vee$**I** (*disjunction intro*)

$$\begin{array}{r|ll} a & P & \\ & & \\ & P \vee Q & a\ \vee\text{I} \end{array}$$

$\vee$**I** (*disjunction intro*)

$$\begin{array}{r|ll} a & P & \\ & & \\ & Q \vee P & a\ \vee\text{I} \end{array}$$

$\vee$**E** (*disjunction exploit*)

$$\begin{array}{r|ll} a & P \vee Q & \\ b & \mid \underline{P} & \text{A } (g,\ a\ \vee\text{E}) \\ & \mid & \\ c & \mid R & \\ d & \mid \underline{Q} & \text{A } (g,\ a\ \vee\text{E}) \\ & \mid & \\ e & \mid R & \\ & R & a,b\text{-}c,d\text{-}e\ \vee\text{E} \end{array}$$

$\supset$**I** (*conditional intro*)

$$\begin{array}{r|ll} a & \mid \underline{P} & \text{A } (g,\ \supset\text{I}) \\ & \mid & \\ b & \mid Q & \\ & P \supset Q & a\text{-}b\ \supset\text{I} \end{array}$$

$\supset$**E** (*conditional exploit*)

$$\begin{array}{r|ll} a & P \supset Q & \\ b & P & \\ & & \\ & Q & a,b\ \supset\text{E} \end{array}$$

$\equiv$**I** (*biconditional intro*)

$$\begin{array}{r|ll} a & \mid \underline{P} & \text{A } (g,\ \equiv\text{I}) \\ & \mid & \\ b & \mid Q & \\ c & \mid \underline{Q} & \text{A } (g,\ \equiv\text{I}) \\ & \mid & \\ d & \mid P & \\ & P \equiv Q & a\text{-}b,c\text{-}d\ \equiv\text{I} \end{array}$$

$\equiv$**E** (*biconditional exploit*)

$$\begin{array}{r|ll} a & P \equiv Q & \\ b & P & \\ & & \\ & Q & a,b\ \equiv\text{E} \end{array}$$

$\equiv$**E** (*biconditional exploit*)

$$\begin{array}{r|ll} a & P \equiv Q & \\ b & Q & \\ & & \\ & P & a,b\ \equiv\text{E} \end{array}$$

NCL $\Gamma \vdash_{NCL} A$ iff there is an *NCL* derivation of A from the members of Γ.

As derived rules, we accept the following "ordinary" and "two-way" rules. The "two-way" rules are usually presented as *replacement* rules. Insofar as we will not have much call to use them that way, in order to streamline demonstrations of soundness, we treat them just as ordinary rules which work in either direction – where it is trivial that the rules are in fact derived in this sense from the rules of *NCL*.

Ordinary Derived Rules

modus tollens

$$\textbf{MT}\ \begin{array}{|l} P \supset Q \\ \neg Q \\ \\ \neg P \end{array}$$

negated biconditional

$$\textbf{NB}\ \begin{array}{|l} P \equiv Q \\ \neg P \\ \\ \neg Q \end{array} \qquad \begin{array}{|l} P \equiv Q \\ \neg Q \\ \\ \neg P \end{array}$$

disjunctive syllogism

$$\textbf{DS}\ \begin{array}{|l} P \vee Q \\ \neg P \\ \\ Q \end{array} \qquad \begin{array}{|l} P \vee Q \\ \neg Q \\ \\ P \end{array}$$

Two-way Derived Rules

DN	$P \triangleleft\triangleright \neg\neg P$	*double negation*
Com	$P \wedge Q \triangleleft\triangleright Q \wedge P$ $P \vee Q \triangleleft\triangleright Q \vee P$	*commutation*
Assoc	$P \wedge (Q \wedge R) \triangleleft\triangleright (P \wedge Q) \wedge R$ $P \vee (Q \vee R) \triangleleft\triangleright (P \vee Q) \vee R$	*association*
Idem	$P \triangleleft\triangleright P \wedge P$ $P \triangleleft\triangleright P \vee P$	*idempotence*
Impl	$P \supset Q \triangleleft\triangleright \neg P \vee Q$ $\neg P \supset Q \triangleleft\triangleright P \vee Q$	*implication*
CDeM	$\neg(P \supset Q) \triangleleft\triangleright P \wedge \neg Q$ $\neg(P \supset \neg Q) \triangleleft\triangleright P \wedge Q$	*Conditional De Morgan*
Trans	$P \supset Q \triangleleft\triangleright \neg Q \supset \neg P$	*transposition*
DeM	$\neg(P \wedge Q) \triangleleft\triangleright \neg P \vee \neg Q$ $\neg(P \vee Q) \triangleleft\triangleright \neg P \wedge \neg Q$	*De Morgan*
Exp	$P \supset (Q \supset R) \triangleleft\triangleright (P \wedge Q) \supset R$	*exportation*
Equiv	$P \equiv Q \triangleleft\triangleright (P \supset Q) \wedge (Q \supset P)$ $P \equiv Q \triangleleft\triangleright (P \wedge Q) \vee (\neg P \wedge \neg Q)$	*equivalence*
Dist	$P \wedge (Q \vee R) \triangleleft\triangleright (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \triangleleft\triangleright (P \vee Q) \wedge (P \vee R)$	*distribution*

Examples. Here are derivations to demonstrate the first form of Impl (among the relatively difficult of derivations for the derived rules).

$\neg P \vee Q \vdash_{NCL} P \supset Q$

1	$\mid \neg P \vee Q$	P
2	$\mid\ \mid \neg P$	A (g, 1 $\vee$E)
3	$\mid\ \mid\ \mid P$	A (g, $\supset$I)
4	$\mid\ \mid\ \mid\ \mid \neg Q$	A (c, $\neg$E)
5	$\mid\ \mid\ \mid\ \mid \neg P$	2 R
6	$\mid\ \mid\ \mid\ \mid P$	3 R
7	$\mid\ \mid\ \mid Q$	4-6 $\neg$E
8	$\mid\ \mid P \supset Q$	3-7 $\supset$I
9	$\mid\ \mid Q$	A (g, 1 $\vee$E)
10	$\mid\ \mid\ \mid P$	A (g, $\supset$I)
11	$\mid\ \mid\ \mid Q$	9 R
12	$\mid\ \mid P \supset Q$	10-11 $\supset$I
13	$\mid P \supset Q$	1,2-8,9-12 $\vee$E

$P \supset Q \vdash_{NCL} \neg P \vee Q$

1	$\mid P \supset Q$	P
2	$\mid\ \mid \neg(\neg P \vee Q)$	A (c, $\neg$E)
3	$\mid\ \mid\ \mid P$	A (c, $\neg$I)
4	$\mid\ \mid\ \mid Q$	1,3 $\supset$E
5	$\mid\ \mid\ \mid \neg P \vee Q$	4 $\vee$I
6	$\mid\ \mid\ \mid \neg(\neg P \vee Q)$	2 R
7	$\mid\ \mid \neg P$	3-6 $\neg$I
8	$\mid\ \mid \neg P \vee Q$	7 $\vee$I
9	$\mid\ \mid \neg(\neg P \vee Q)$	2 R
10	$\mid \neg P \vee Q$	2-9 $\neg$E

1.3 Soundness and Completeness

The following are standard arguments. Cases that are omitted are like ones worked, and so left to the reader.

THEOREM 1.1 *NCL is sound: If* $\Gamma \vdash_{NCL} A$ *then* $\Gamma \models_{CL} A$.

L1.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models_{CL} P$, then $\Gamma' \models_{CL} P$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \models_{CL} P$, but $\Gamma' \not\models_{CL} P$. From the latter, by VCL, there is some v such that $v(\Gamma') = 1$ but $v(P) = 0$. But since $v(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $v(\Gamma) = 1$; so v is a *CL* interpretation such that $v(\Gamma) = 1$ but $v(P) = 0$; so by VCL, $\Gamma \not\models_{CL} P$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models_{CL} P$, then $\Gamma' \models_{CL} P$.

Main result: For each line in a derivation let A_i be the formula on line i and set Γ_i equal to the set of all premises and assumptions whose scope includes line i. Suppose $\Gamma \vdash_{NCL} A$. Then there is a derivation of A from premises in Γ where A appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models_{CL} A_i$. The case when $A_i = A$ is the desired result.

Basis: A_1 is a premise or an assumption. Then $\Gamma_1 = \{A_1\}$; so $v(\Gamma_1) = 1$ iff $v(A_1) = 1$; so there is no v such that $v(\Gamma_1) = 1$ but $v(A_1) = 0$. So by VCL, $\Gamma_1 \models_{CL} A_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models_{CL} A_i$.

Show: $\Gamma_k \models_{CL} A_k$.

A_k is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I or $\equiv$E. If A_k is a premise or an assumption, then as in the basis, $\Gamma_k \models_{CL} A_k$. So suppose A_k arises by one of the rules.

(R)

($\supset$I) If A_k arises by $\supset$I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & \underline{P} \\
 & & \\
j & & Q \\
k & \multicolumn{2}{l}{P \supset Q}
\end{array}$$

where $j < k$ and A_k is $P \supset Q$. By assumption, $\Gamma_j \models_{CL} Q$; and by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{P\}$; so by L1.1, $\Gamma_k \cup \{P\} \models_{CL} Q$. Suppose $\Gamma_k \not\models_{CL} P \supset Q$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(P \supset Q) = 0$; from the latter, by TCL($\supset$), $v(P) = 1$ and $v(Q) = 0$; so $v(\Gamma_k) = 1$ and $v(P) = 1$; so $v(\Gamma_k \cup \{P\}) = 1$; so by VCL, $v(Q) = 1$. This is impossible; reject the assumption: $\Gamma_k \models_{CL} P \supset Q$, which is to say, $\Gamma_k \models_{CL} A_k$.

($\supset$E) If A_k arises by $\supset$E, then the picture is like this,

$$\begin{array}{r|l}
i & P \supset Q \\
j & P \\
 & \\
k & Q
\end{array}$$

where $i, j < k$ and A_k is Q. By assumption, $\Gamma_i \models_{CL} P \supset Q$ and $\Gamma_j \models_{CL} P$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L1.1, $\Gamma_k \models_{CL} P \supset Q$ and $\Gamma_k \models_{CL} P$. Suppose $\Gamma_k \not\models_{CL} Q$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(Q) = 0$; since $v(\Gamma_k) = 1$, by VCL, $v(P \supset Q) = 1$ and $v(P) = 1$; from the former, by TCL($\supset$), $v(P) = 0$ or $v(Q) = 1$; so $v(Q) = 1$. This is impossible; reject the assumption: $\Gamma_k \models_{CL} Q$, which is to say, $\Gamma_k \models_{CL} A_k$.

($\wedge$I)

($\wedge$E)

(¬I) If A_k arises by ¬I, then the picture is like this,

$$\begin{array}{r|c|l}
 & & \underline{P} \\
 & & \\
i & & Q \\
j & & \neg Q \\
k & \multicolumn{2}{l}{\neg P}
\end{array}$$

where $i, j < k$ and A_k is $\neg P$. By assumption, $\Gamma_i \models_{CL} Q$ and $\Gamma_j \models_{CL} \neg Q$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{P\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{P\}$; so by L1.1, $\Gamma_k \cup \{P\} \models_{CL} Q$ and $\Gamma_k \cup \{P\} \models_{CL} \neg Q$. Suppose $\Gamma_k \not\models_{CL} \neg P$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(\neg P) = 0$; from the latter, by TCL(¬), $v(P) = 1$; so $v(\Gamma_k) = 1$ and $v(P) = 1$; so $v(\Gamma_k \cup \{P\}) = 1$; so by VCL, $v(Q) = 1$ and $v(\neg Q) = 1$; from the latter, by TCL(¬), $v(Q) = 0$. This is impossible; reject the assumption: $\Gamma_k \models_{CL} \neg P$, which is to say, $\Gamma_k \models_{CL} A_k$.

(¬E)

(∨I) If A_k arises by ∨I, then the picture is like this,

$$\begin{array}{r|l}
j & P \\
 & \\
k & P \vee Q
\end{array}
\qquad\qquad \text{or} \qquad\qquad
\begin{array}{r|l}
j & P \\
 & \\
k & Q \vee P
\end{array}$$

where $j < k$ and A_k is $P \vee Q$ or $Q \vee P$. Consider the first case. By assumption, $\Gamma_j \models_{CL} P$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k$; so by L1.1, $\Gamma_k \models_{CL} P$. Suppose $\Gamma_k \not\models_{CL} P \vee Q$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(P \vee Q) = 0$; since $v(\Gamma_k) = 1$, by VCL, $v(P) = 1$; but since $v(P \vee Q) = 0$, by TCL(∨), $v(P) = 0$ and $v(Q) = 0$. This is impossible; reject the assumption: $\Gamma_k \models_{CL} P \vee Q$, which is to say, $\Gamma_k \models_{CL} A_k$. And similarly when A_k is $Q \vee P$.

(∨E) If A_k arises by ∨E, then the picture is like this,

$$\begin{array}{r|c|l}
h & \multicolumn{2}{l}{P \vee Q} \\
 & & \underline{P} \\
 & & \\
i & & R \\
 & \multicolumn{2}{l}{} \\
 & & \underline{Q} \\
 & & \\
j & & R \\
k & \multicolumn{2}{l}{R}
\end{array}$$

where $h, i, j < k$ and A_k is R. By assumption, $\Gamma_h \models_{CL} P \vee Q$, $\Gamma_i \models_{CL} R$ and $\Gamma_j \models_{CL} R$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k \cup \{P\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{Q\}$; so by L1.1, $\Gamma_k \models_{CL} P \vee Q$, $\Gamma_k \cup \{P\} \models_{CL} R$ and $\Gamma_k \cup \{Q\} \models_{CL} R$. Suppose $\Gamma_k \not\models_{CL} R$; then by VCL, there is some v such that $v(\Gamma_k) = 1$ but $v(R) = 0$. Since $v(\Gamma_k) = 1$, by VCL, $v(P \vee Q) = 1$; so by TCL($\vee$), $v(P) = 1$ or $v(Q) = 1$. Suppose, for the moment, that $v(P) = 1$; then $v(\Gamma_k) = 1$ and $v(P) = 1$; so $v(\Gamma_k \cup \{P\}) = 1$; so by VCL, $v(R) = 1$; this is impossible; reject the assumption: $v(P) \neq 1$; so $v(Q) = 1$; so $v(\Gamma_k) = 1$ and $v(Q) = 1$; so $v(\Gamma_k \cup \{Q\}) = 1$; so by VCL, $v(R) = 1$; this is impossible; reject the assumption: $\Gamma_k \models_{CL} R$, which is to say, $\Gamma_k \models_{CL} A_k$.

($\equiv$I)

($\equiv$E)

For any i, $\Gamma_i \models_{CL} A_i$.

THEOREM 1.2 *NCL is complete: if* $\Gamma \models_{CL} A$ *then* $\Gamma \vdash_{NCL} A$.

CON Γ is CONSISTENT iff there is no A such that $\Gamma \vdash_{NCL} A$ and $\Gamma \vdash_{NCL} \neg A$.

L1.2 If $\Gamma \nvdash_{NCL} \neg P$, then $\Gamma \cup \{P\}$ is consistent.

Suppose $\Gamma \nvdash_{NCL} \neg P$ but $\Gamma \cup \{P\}$ is inconsistent. Then there is some A such that $\Gamma \cup \{P\} \vdash_{NCL} A$ and $\Gamma \cup \{P\} \vdash_{NCL} \neg A$. But then we can argue,

1	Γ	
2	$\quad P$	A (c, $\neg$I)
3	$\quad A$	from $\Gamma \cup \{P\}$
4	$\quad \neg A$	from $\Gamma \cup \{P\}$
5	$\neg P$	2-4 $\neg$I

So $\Gamma \vdash_{NCL} \neg P$. But this is impossible; reject the assumption: if $\Gamma \nvdash_{NCL} \neg P$, then $\Gamma \cup \{P\}$ is consistent.

L1.3 There is an enumeration of all the formulas, $A_1, A_2 \ldots$

Proof by construction in the usual way.[1]

MAX Γ is MAXIMAL iff for any A either $\Gamma \vdash_{NCL} A$ or $\Gamma \vdash_{NCL} \neg A$.

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[1] For this, and extended discussion of the larger argument, see e.g. [6, §11.2].

C(Γ') We construct a Γ' from Γ as follows. Set $\Omega_0 = \Gamma$. By L1.3, there is an enumeration, $A_1, A_2 \ldots$ of all the formulas; for any A_i in this series set,

$$\begin{array}{ll} \Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \vdash_{NCL} \neg A_i \\ \Omega_i = \Omega_{i-1} \cup \{A_i\} & \text{if } \Omega_{i-1} \nvdash_{NCL} \neg A_i \end{array}$$

then

$$\Gamma' = \bigcup_{i \geq 0} \Omega_i$$

L1.4 Γ' is maximal.

Suppose Γ' is not maximal. Then there is some A_i such that $\Gamma' \nvdash_{NCL} A_i$ and $\Gamma' \nvdash_{NCL} \neg A_i$. Whatever i may be, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash_{NCL} \neg A_i$ then $\Gamma' \vdash_{NCL} \neg A_i$; but $\Gamma' \nvdash_{NCL} \neg A_i$; so $\Omega_{i-1} \nvdash_{NCL} \neg A_i$; so by construction, $\Omega_i = \Omega_{i-1} \cup \{A_i\}$; so by construction, $A_i \in \Gamma'$; so $\Gamma' \vdash_{NCL} A_i$. This is impossible; reject the assumption: Γ' is maximal.

L1.5 If Γ is consistent, then each Ω_i is consistent.

Suppose Γ is consistent.

Basis: $\Omega_0 = \Gamma$ and Γ is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either Ω_{k-1} or $\Omega_{k-1} \cup \{A_k\}$. Suppose the former; by assumption, Ω_{k-1} is consistent; so Ω_k is consistent. Suppose the latter; then by construction, $\Omega_{k-1} \nvdash_{NCL} \neg A_k$; so by L1.2, $\Omega_{k-1} \cup \{A_k\}$ is consistent; so Ω_k is consistent.

For any i, Ω_i is consistent.

L1.6 If Γ is consistent, then Γ' is consistent.

Suppose Γ is consistent, but Γ' is not; from the latter, there is some P such that $\Gamma' \vdash_{NCL} P$ and $\Gamma' \vdash_{NCL} \neg P$. Consider derivations D1 and D2 of these results and the premises $A_i \ldots A_j$ of these derivations. Where A_j is the last of these premises in the enumeration of formulas, by the construction of Γ', each of $A_i \ldots A_j$ must be a member of Ω_j; so D1 and D2 are derivations from Ω_j; so Ω_j is not consistent. But since Γ is consistent, by L1.5, Ω_j is consistent. This is impossible; reject the assumption: if Γ is consistent then Γ' is consistent.

C(v) We construct a *CL* interpretation v based on Γ' as follows. For any parameter p, set $v(p) = 1$ iff $\Gamma' \vdash_{NCL} p$.

L1.7 If Γ is consistent then for any A, $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

Suppose Γ is consistent. By L1.4, Γ' is maximal; by L1.6, Γ' is consistent. Now by induction on the number of operators in A,

Basis: If A has no operators, then it is a parameter p and by construction, $v(p) = 1$ iff $\Gamma' \vdash_{NCL} p$. So $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

Assp: For any i, $0 \leq i < k$, if A has i operators, then $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

Show: If A has k operators, then $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

If A has k operators, then it is of the form $\neg P$, $P \supset Q$, $P \wedge Q$, $P \vee Q$ or $P \equiv Q$ where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. (i) Suppose $v(A) = 1$; then $v(\neg P) = 1$; so by TCL($\neg$), $v(P) = 0$; so by assumption, $\Gamma' \nvdash_{NCL} P$; so by maximality, $\Gamma' \vdash_{NCL} \neg P$, where this is to say, $\Gamma' \vdash_{NCL} A$. (ii) Suppose $\Gamma' \vdash_{NCL} A$; then $\Gamma' \vdash_{NCL} \neg P$; so by consistency, $\Gamma' \nvdash_{NCL} P$; so by assumption, $v(P) = 0$; so by TCL($\neg$), $v(\neg P) = 1$, where this is to say, $v(A) = 1$. So $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

($\supset$) A is $P \supset Q$. (i) Suppose $v(A) = 1$ but $\Gamma' \nvdash_{NCL} A$; then $v(P \supset Q) = 1$ but $\Gamma' \nvdash_{NCL} P \supset Q$. From the latter, by maximality, $\Gamma' \vdash_{NCL} \neg(P \supset Q)$; from this it follows, by simple derivations, that $\Gamma' \vdash_{NCL} P$ and $\Gamma' \vdash_{NCL} \neg Q$; so by consistency, $\Gamma' \nvdash_{NCL} Q$; so by assumption, $v(P) = 1$ and $v(Q) = 0$; so by TCL($\supset$), $v(P \supset Q) = 0$. This is impossible; reject the assumption: if $v(A) = 1$ then $\Gamma' \vdash_{NCL} A$.

(ii) Suppose $\Gamma' \vdash_{NCL} A$ but $v(A) = 0$; then $\Gamma' \vdash_{NCL} P \supset Q$ but $v(P \supset Q) = 0$. From the latter, by TCL($\supset$), $v(P) = 1$ and $v(Q) = 0$; so by assumption, $\Gamma' \vdash_{NCL} P$ and $\Gamma' \nvdash_{NCL} Q$; but since $\Gamma' \vdash_{NCL} P \supset Q$ and $\Gamma' \vdash_{NCL} P$, by ($\supset$E), $\Gamma' \vdash_{NCL} Q$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NCL} A$, then $v(A) = 1$. So $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

($\wedge$)

($\vee$)

($\equiv$)

For any A, $v(A) = 1$ iff $\Gamma' \vdash_{NCL} A$.

L1.8 If Γ is consistent, then $v(\Gamma) = 1$.

Suppose Γ is consistent and $A \in \Gamma$; then by construction, $A \in \Gamma'$; so $\Gamma' \vdash_{\mathit{NCL}} A$; so since Γ is consistent, by L1.7, $v(A) = 1$. And similarly for any $A \in \Gamma$. So $v(\Gamma) = 1$.

Main result: Suppose $\Gamma \vDash_{\mathit{CL}} A$ but $\Gamma \nvdash_{\mathit{NCL}} A$. By (DN), if $\Gamma \vdash_{\mathit{NCL}} \neg\neg A$, then $\Gamma \vdash_{\mathit{NCL}} A$; so $\Gamma \nvdash_{\mathit{NCL}} \neg\neg A$; so by L1.2, $\Gamma \cup \{\neg A\}$ is consistent; so by L1.8, there is a v constructed as above such that $v(\Gamma \cup \{\neg A\}) = 1$; so $v(\neg A) = 1$; so by TCL($\neg$), $v(A) = 0$; so $v(\Gamma) = 1$ and $v(A) = 0$; so by VCL, $\Gamma \nvDash_{\mathit{CL}} A$. This is impossible; reject the assumption: if $\Gamma \vDash_{\mathit{CL}} A$, then $\Gamma \vdash_{\mathit{NCL}} A$.

2 Normal Modal Logics: $\boldsymbol{K\alpha}$, $\boldsymbol{K^t_\alpha}$ (ch. 2,3)

2.1 Language / Semantic Notions

LK$^{(t)}_\alpha$ Allow K$^{(t)}_\alpha$ to be either Kα or K$^t_\alpha$, depending on context, where for both Kα and K$^t_\alpha$ systems, the VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, and $\equiv$; along with $\Box$ and $\Diamond$ for Kα systems; but with $[\text{F}]$, $\langle\text{F}\rangle$, $[\text{P}]$, and $\langle\text{P}\rangle$ for K$^t_\alpha$ systems. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$, $\Diamond A$, $[\text{F}]A$, $\langle\text{F}\rangle A$, $[\text{P}]A$, and $\langle\text{P}\rangle A$.

IK$^{(t)}_\alpha$ For any of these systems except Kv, an INTERPRETATION is a triple $\langle W, R, v\rangle$ where W is a set of worlds, R is a subset of $W^2 = W \times W$, and v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$. For $x, y, z \in W$, where α is empty or indicates some combination of the following constraints,

η	For any x, there is a y such that xRy	extendability
ρ	for all x, xRx	reflexivity
σ	for all x, y, if xRy then yRx	symmetry
τ	for all x, y, z, if xRy and yRz then xRz	transitivity
η'	For any x, there is a y such that yRx	backward extendibility
δ	If xRy then for some z, xRz and zRy	denseness
φ	If xRy and xRz then yRz or $y = z$ or zRy	forward convergence
β	If yRx and zRx then yRz or $y = z$ or zRy	backward convergence

$\langle W, R, v\rangle$ is a K$^{(t)}_\alpha$ interpretation when R meets the constraints from α.

TK For complex expressions,

(¬) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

(∧) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

(∨) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

(⊃) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

(≡) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

For K,

(◊) $v_w(\Diamond A) = 1$ if some $x \in W$ such that wRx has $v_x(A) = 1$, and 0 otherwise.

(□) $v_w(\Box A) = 1$ if all $x \in W$ such that wRx have $v_x(A) = 1$, and 0 otherwise.

For K^t,

([F]) $v_w([\mathrm{F}]A) = 1$ iff all $x \in W$ such that wRx have $v_x(A) = 1$.

([P]) $v_w([\mathrm{P}]A) = 1$ iff all $x \in W$ such that xRw have $v_x(A) = 1$.

(⟨F⟩) $v_w(\langle\mathrm{F}\rangle A) = 1$ iff some $x \in W$ such that wRx has $v_x(A) = 1$.

(⟨P⟩) $v_w(\langle\mathrm{P}\rangle A) = 1$ iff some $x \in W$ such that xRw has $v_x(A) = 1$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VK$_\alpha^{(t)}$ $\Gamma \models_{K_\alpha^{(t)}} A$ iff there is no K$_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

System Kv. For Kv either accept the constraint, (v) for all x, y, xRy. Then let everything work as before. Otherwise simplify the semantics: An interpretation is just $\langle W, v\rangle$. For TK(□) and TK(◊) substitute,

TK (◊)$_v$ $v_w(\Diamond A) = 1$ iff for some $x \in W$, $v_x(A) = 1$.

(□)$_v$ $v_w(\Box A) = 1$ iff for all $x \in W$, $v_x(A) = 1$.

then,

VKv $\Gamma \models_{Kv} A$ iff there is no Kv interpretation $\langle W, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

2.2 Natural Derivations: $\boldsymbol{NK}_{\alpha}^{(t)}$

Where s is any integer, let A_s be a SUBSCRIPTED FORMULA. For subscripts s and t allow also expressions of the sort, $s.t$. As in Priest, intuitively, subscripts indicate worlds, where A_s is true or false at world s, and $s.t$ just in case world s has access to world t. Derivation rules apply to these expressions. Rules for ¬, ∧, ∨, ⊃, and ≡ are like ones from before, but with consistent subscripts. Rules for □, ◊, [F], ⟨F⟩, [P], and ⟨P⟩ are new.[2]

$$\begin{array}{r|l} \textbf{R} & P_s \\ & \\ & P_s \end{array}$$

$$\begin{array}{r|l} \neg\textbf{I} & \begin{array}{|l} \underline{P_s} \\ \\ Q_t \\ \neg Q_t \end{array} \\ & \neg P_s \end{array}$$

$$\begin{array}{r|l} \neg\textbf{E} & \begin{array}{|l} \underline{\neg P_s} \\ \\ Q_t \\ \neg Q_t \end{array} \\ & P_s \end{array}$$

$$\begin{array}{r|l} \wedge\textbf{I} & P_s \\ & Q_s \\ & \\ & (P \wedge Q)_s \end{array}$$

$$\begin{array}{r|l} \wedge\textbf{E} & (P \wedge Q)_s \\ & \\ & P_s \end{array}$$

$$\begin{array}{r|l} \wedge\textbf{E} & (P \wedge Q)_s \\ & \\ & Q_s \end{array}$$

$$\begin{array}{r|l} \vee\textbf{I} & P_s \\ & \\ & (P \vee Q)_s \end{array}$$

$$\begin{array}{r|l} \vee\textbf{I} & P_s \\ & \\ & (Q \vee P)_s \end{array}$$

$$\begin{array}{r|l} \vee\textbf{E} & (P \vee Q)_s \\ & \begin{array}{|l} \underline{P_s} \\ \\ R_t \end{array} \\ & \begin{array}{|l} \underline{Q_s} \\ \\ R_t \end{array} \\ & R_t \end{array}$$

$$\begin{array}{r|l} \supset\textbf{I} & \begin{array}{|l} \underline{P_s} \\ \\ Q_s \end{array} \\ & (P \supset Q)_s \end{array}$$

$$\begin{array}{r|l} \supset\textbf{E} & (P \supset Q)_s \\ & P_s \\ & \\ & Q_s \end{array}$$

$$\begin{array}{r|l} \equiv\textbf{I} & \begin{array}{|l} \underline{P_s} \\ \\ Q_s \end{array} \\ & \begin{array}{|l} \underline{Q_s} \\ \\ P_s \end{array} \\ & (P \equiv Q)_s \end{array}$$

$$\begin{array}{r|l} \equiv\textbf{E} & (P \equiv Q)_s \\ & P_s \\ & \\ & Q_s \end{array}$$

$$\begin{array}{r|l} \equiv\textbf{E} & (P \equiv Q)_s \\ & Q_s \\ & \\ & P_s \end{array}$$

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[2] There is no uniformity about how to do natural deduction in modal logic. Most avoid subscripts altogether. Another option uses subscripts of the sort $i.j \ldots k$ (cf. prefixes on tableaux in [2]); the result is elegant, but not so flexible as this account inspired by Priest, and we will need the flexibility, as we approach increasingly complex systems.

For K,

$$\textbf{$\Box$I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ \\ P_t \end{array} \\ \Box P_s \end{array} \qquad \textbf{$\Box$E}\ \begin{array}{|l} \Box P_s \\ s.t \\ \\ P_t \end{array} \qquad \textbf{$\Diamond$I}\ \begin{array}{|l} P_t \\ s.t \\ \\ \Diamond P_s \end{array} \qquad \textbf{$\Diamond$E}\ \begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} s.t \\ P_t \\ \hline \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

$\Box$I: where t does not appear in any undischarged premise or assumption

$\Diamond$E: where t does not appear in any undischarged premise or assumption and is not u

For K^t,

$$\textbf{[F]I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ \\ P_t \end{array} \\ [\mathrm{F}]P_s \end{array} \qquad \textbf{[F]E}\ \begin{array}{|l} [\mathrm{F}]P_s \\ s.t \\ \\ P_t \end{array} \qquad \langle\textbf{F}\rangle\textbf{I}\ \begin{array}{|l} P_t \\ s.t \\ \\ \langle\mathrm{F}\rangle P_s \end{array} \qquad \langle\textbf{F}\rangle\textbf{E}\ \begin{array}{|l} \langle\mathrm{F}\rangle P_s \\ \begin{array}{|l} s.t \\ P_t \\ \hline \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

[F]I: where t does not appear in any undischarged premise or assumption

$\langle$F$\rangle$E: where t does not appear in any undischarged premise or assumption and is not u

$$\textbf{[P]I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ \\ P_s \end{array} \\ [\mathrm{P}]P_t \end{array} \qquad \textbf{[P]E}\ \begin{array}{|l} [\mathrm{P}]P_t \\ s.t \\ \\ P_s \end{array} \qquad \langle\textbf{P}\rangle\textbf{I}\ \begin{array}{|l} P_s \\ s.t \\ \\ \langle\mathrm{P}\rangle P_t \end{array} \qquad \langle\textbf{P}\rangle\textbf{E}\ \begin{array}{|l} \langle\mathrm{P}\rangle P_t \\ \begin{array}{|l} s.t \\ P_s \\ \hline \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

[P]I: where s does not appear in any undischarged premise or assumption

$\langle$P$\rangle$E: where s does not appear in any undischarged premise or assumption and is not u

These are the rules of $NK^{(t)}$. Other systems $NK^{(t)}_{\alpha}$ add from the following, for *access manipulation*, according to constraints in α. Where $\mathcal{A}(i)$ is any expression in which i appears, and $\mathcal{A}(j)$ is the same expression with j substituted for i,

$$\textbf{AM}\eta\ \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ \\ P_u \end{array} \\ P_u \end{array} \qquad \textbf{AM}\rho\ \begin{array}{|l} \\ \\ \\ s.s \end{array} \qquad \textbf{AM}\sigma\ \begin{array}{|l} s.t \\ \\ \\ t.s \end{array}$$

AMη: where t does not appear in any undischarged premise or assumption and is not u

$$\mathbf{AM}\tau \quad \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

$$\mathbf{AM}\eta' \quad \begin{array}{|l} \begin{array}{|l} \underline{s.t} \\ \\ P_u \end{array} \\ P_u \end{array}$$

where s does not appear in any undischarged premise or assumption and is not u

$$\mathbf{AM}\delta \quad \begin{array}{|l} s.t \\ \begin{array}{|l} s.a \\ \underline{a.t} \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

where a does not appear in any undischarged premise or assumption and is not u

$$\mathbf{=E} \quad \begin{array}{|l} s = t \\ \mathcal{A}(s) \\ \\ \mathcal{A}(t) \end{array} \qquad \begin{array}{|l} t = s \\ \mathcal{A}(s) \\ \\ \mathcal{A}(t) \end{array}$$

$$\mathbf{AM}\varphi \quad \begin{array}{|l} r.s \\ r.t \\ \begin{array}{|l} \underline{s.t} \\ \\ Q_u \end{array} \\ \begin{array}{|l} \underline{s = t} \\ \\ Q_u \end{array} \\ \begin{array}{|l} \underline{t.s} \\ \\ Q_u \end{array} \\ Q_u \end{array} \qquad \mathbf{AM}\beta \quad \begin{array}{|l} s.r \\ t.r \\ \begin{array}{|l} \underline{s.t} \\ \\ Q_u \end{array} \\ \begin{array}{|l} \underline{s = t} \\ \\ Q_u \end{array} \\ \begin{array}{|l} \underline{t.s} \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

AMρ has no premise. In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for $\Box$I, $\Diamond$E, [F]I, $\langle$F$\rangle$E, [P]I, $\langle$P$\rangle$E, AMη, AMη', AMδ, AMφ, or AMβ. Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then,

NK$^{(t)}_{\alpha}$ $\Gamma \vdash_{NK^{(t)}_{\alpha}} A$ iff there is an $NK^{(t)}_{\alpha}$ derivation of A_0 from the members of Γ_0.

Derived rules carry over from *NCL* as one would expect, with subscripts constant throughout. Thus, e.g.,

$$\mathbf{MT} \quad \begin{array}{|l} (P \supset Q)_s \\ \neg Q_s \\ \\ \neg P_s \end{array} \qquad\qquad \mathbf{Impl} \quad \begin{array}{l} (P \supset Q)_s \ \lhd\rhd\ (\neg P \vee Q)_s \\ (\neg P \supset Q)_s \ \lhd\rhd\ (P \vee Q)_s \end{array}$$

Allow also the additional rule for *modal negation* and *tense modal negation*,

$$\begin{array}{llll}
\textbf{MN} & \Box P_s \;\lhd\rhd\; \neg\Diamond\neg P_s & & \neg\Box P_s \;\lhd\rhd\; \Diamond\neg P_s \\
 & \Diamond P_s \;\lhd\rhd\; \neg\Box\neg P_s & & \neg\Diamond P_s \;\lhd\rhd\; \Box\neg P_s \\
 & & & \\
\textbf{TMN} & [\mathrm{F}]P_s \;\lhd\rhd\; \neg\langle\mathrm{F}\rangle\neg P_s & & \neg[\mathrm{F}]P_s \;\lhd\rhd\; \langle\mathrm{F}\rangle\neg P_s \\
 & \langle\mathrm{F}\rangle P_s \;\lhd\rhd\; \neg[\mathrm{F}]\neg P_s & & \neg\langle\mathrm{F}\rangle P_s \;\lhd\rhd\; [\mathrm{F}]\neg P_s \\
 & [\mathrm{P}]P_s \;\lhd\rhd\; \neg\langle\mathrm{P}\rangle\neg P_s & & \neg[\mathrm{P}]P_s \;\lhd\rhd\; \langle\mathrm{P}\rangle\neg P_s \\
 & \langle\mathrm{P}\rangle P_s \;\lhd\rhd\; \neg[\mathrm{P}]\neg P_s & & \neg\langle\mathrm{P}\rangle P_s \;\lhd\rhd\; [\mathrm{P}]\neg P_s
\end{array}$$

System NKυ. For $NK\upsilon$, eliminate expressions of the sort $s.t$ and rules for access manipulation. Let $\top$ be an arbitrary tautology (say, $p \supset p$). Then for $\Box$I, $\Box$E, $\Diamond$I and $\Diamond$E, substitute,

$$\begin{array}{l|l}
\Box\mathbf{I}\upsilon & \mid \underline{\top_t} \\
 & \mid \\
 & \mid P_t \\
 & \Box P_s
\end{array}
\qquad
\begin{array}{l|l}
\Box\mathbf{E}\upsilon & \Box P_s \\
 & \\
 & P_t
\end{array}
\qquad
\begin{array}{l|l}
\Diamond\mathbf{I}\upsilon & P_t \\
 & \\
 & \Diamond P_s
\end{array}
\qquad
\begin{array}{l|l}
\Diamond\mathbf{E}\upsilon & \Diamond P_s \\
 & \mid \underline{P_t} \\
 & \mid \\
 & \mid Q_u \\
 & Q_u
\end{array}$$

where t does not appear in any undischarged premise or assumption

where t does not appear in any undischarged premise or assumption and is not u

Examples. Here are derivations to exhibit left-hand forms of the rule for modal negation as derived in NK (and so any $NK\alpha$).

$\neg\Diamond\neg P \vdash_{NK} \Box P$

$$\begin{array}{r|lll}
1 & \underline{\neg\Diamond\neg P_0} & & \text{P} \\
2 & \mid \underline{0.1} & & \text{A } (g, \Box\text{I}) \\
3 & \mid \mid \underline{\neg P_1} & & \text{A } (c, \neg\text{E}) \\
4 & \mid \mid \Diamond\neg P_0 & & 2{,}3\ \Diamond\text{I} \\
5 & \mid \mid \neg\Diamond\neg P_0 & & 1\ \text{R} \\
6 & \mid P_1 & & 3\text{-}5\ \neg\text{E} \\
7 & \Box P_0 & & 2\text{-}6\ \Box\text{I}
\end{array}$$

$\Box P \vdash_{NK} \neg\Diamond\neg P$

$$\begin{array}{r|lll}
1 & \underline{\Box P_0} & & \text{P} \\
2 & \mid \underline{\Diamond\neg P_0} & & \text{A } (c, \neg\text{I}) \\
3 & \mid \mid 0.1 & & \text{A } (g, 2\ \Diamond\text{E}) \\
4 & \mid \mid \underline{\neg P_1} & & \\
5 & \mid \mid \mid \underline{\Diamond\neg P_0} & & \text{A } (c, \neg\text{I}) \\
6 & \mid \mid \mid \neg P_1 & & 4\ \text{R} \\
7 & \mid \mid \mid P_1 & & 1{,}3\ \Box\text{E} \\
8 & \mid \mid \neg\Diamond\neg P_0 & & 5\text{-}7\ \neg\text{I} \\
9 & \mid \neg\Diamond\neg P_0 & & 2{,}3\text{-}8\ \Diamond\text{E} \\
10 & \mid \Diamond\neg P_0 & & 2\ \text{R} \\
11 & \neg\Diamond\neg P_0 & & 2\text{-}10\ \neg\text{I}
\end{array}$$

$\neg\Box\neg P \vdash_{NK} \Diamond P$

1	$\mid \neg\Box\neg P_0$	P
2	$\mid\mid \neg\Diamond P_0$	A (c, ¬E)
3	$\mid\mid\mid 0.1$	A (g, □I)
4	$\mid\mid\mid\mid P_1$	A (c, ¬I)
5	$\mid\mid\mid\mid \Diamond P_0$	3,4 ◇I
6	$\mid\mid\mid\mid \neg\Diamond P_0$	2 R
7	$\mid\mid\mid \neg P_1$	4-6 ¬I
8	$\mid\mid \Box\neg P_0$	3-7 □I
9	$\mid\mid \neg\Box\neg P_0$	1 R
10	$\mid \Diamond P_0$	2-9 ¬E

$\Diamond P \vdash_{NK} \neg\Box\neg P$

1	$\mid \Diamond P_0$	P
2	$\mid\mid 0.1$	A (g, 1 ◇E)
3	$\mid\mid P_1$	
4	$\mid\mid\mid \Box\neg P_0$	A (c, ¬I)
5	$\mid\mid\mid \neg P_1$	2,4 □E
6	$\mid\mid\mid P_1$	3 R
7	$\mid\mid \neg\Box\neg P_0$	4-6 ¬I
8	$\mid \neg\Box\neg P_0$	1,2-7 ◇E

And some derivations in some of the other other systems,

$\vdash_{NK\eta} \Box P \supset \Diamond P$

1	$\mid\mid \Box P_0$	A (g, ⊃I)
2	$\mid\mid\mid 0.1$	A (g, AMη)
3	$\mid\mid\mid P_1$	1,2 □E
4	$\mid\mid\mid \Diamond P_0$	2,3 ◇I
5	$\mid\mid \Diamond P_0$	2-4 AMη
6	$\mid (\Box P \supset \Diamond P)_0$	1-5 ⊃I

$\vdash_{NK\rho} \Box P \supset P$

1	$\mid\mid \Box P_0$	A (g, ⊃I)
2	$\mid\mid 0.0$	AMρ
3	$\mid\mid P_0$	1,2 □E
4	$\mid (\Box P \supset P)_0$	1-3 ⊃I

$\vdash_{NK\sigma} P \supset \Box\Diamond P$

1	$\mid\mid P_0$	A (g, ⊃I)
2	$\mid\mid\mid 0.1$	A (g, □I)
3	$\mid\mid\mid 1.0$	2 AMσ
4	$\mid\mid\mid \Diamond P_1$	1,3 ◇I
5	$\mid\mid \Box\Diamond P_0$	2-4 □I
6	$\mid (P \supset \Box\Diamond P)_0$	1-5 ⊃I

$\vdash_{NK\tau} \Box P \supset \Box\Box P$

1	$\mid\mid \Box P_0$	A (g, ⊃I)
2	$\mid\mid\mid 0.1$	A (g, □I)
3	$\mid\mid\mid\mid 1.2$	A (g, □I)
4	$\mid\mid\mid\mid 0.2$	2,3 AMτ
5	$\mid\mid\mid\mid P_2$	1,4 □E
6	$\mid\mid\mid \Box P_1$	3-5 □I
7	$\mid\mid \Box\Box P_0$	2-6 □I
8	$\mid (\Box P \supset \Box\Box P)_0$	1-7 ⊃I

$\vdash_{NK\sigma\tau} \Diamond P \supset \Box\Diamond P$

1	$\mid\ \mid\ \underline{\Diamond P_0}$	A (g, $\supset$I)
2	$\mid\ \mid\ \mid\ 0.1$	A (g, 1 $\Diamond$E)
3	$\mid\ \mid\ \mid\ \underline{P_1}$	
4	$\mid\ \mid\ \mid\ \mid\ \underline{0.2}$	A (g, $\Box$I)
5	$\mid\ \mid\ \mid\ \mid\ 2.0$	4 AMσ
6	$\mid\ \mid\ \mid\ \mid\ 2.1$	5,2 AMτ
7	$\mid\ \mid\ \mid\ \mid\ \Diamond P_2$	3,6 $\Diamond$I
8	$\mid\ \mid\ \mid\ \Box\Diamond P_0$	4-7 $\Box$I
9	$\mid\ \mid\ \Box\Diamond P_0$	1,2-8 $\Diamond$E
10	$\mid\ (\Diamond P \supset \Box\Diamond P)_0$	1-9 $\supset$I

$\vdash_{NK\upsilon} \Diamond P \supset \Box\Diamond P$

1	$\mid\ \mid\ \underline{\Diamond P_0}$	A (g, $\supset$I)
2	$\mid\ \mid\ \mid\ \underline{P_1}$	A (g, 1 $\Diamond$E)
3	$\mid\ \mid\ \mid\ \mid\ \underline{\top_2}$	A (g, $\Box$I)
4	$\mid\ \mid\ \mid\ \mid\ \Diamond P_2$	2 $\Diamond$I
5	$\mid\ \mid\ \mid\ \Box\Diamond P_0$	3-4 $\Box$I
6	$\mid\ \mid\ \Box\Diamond P_0$	1,2-5 $\Diamond$E
7	$\mid\ (\Diamond P \supset \Box\Diamond P)_0$	1-6 $\supset$I

$[\mathrm{P}][\mathrm{P}]A \vdash_{NK^t_\delta} [\mathrm{P}]A$

1	$\mid\ \underline{[\mathrm{P}][\mathrm{P}]A_0}$	P
2	$\mid\ \mid\ \underline{1.0}$	A (g, [P]I)
3	$\mid\ \mid\ \mid\ 1.2$	A (g, AMδ)
4	$\mid\ \mid\ \mid\ \underline{2.0}$	
5	$\mid\ \mid\ \mid\ [\mathrm{P}]A_2$	1,4 [P]E
6	$\mid\ \mid\ \mid\ A_1$	5,3 [P]E
7	$\mid\ \mid\ A_1$	2,3-6 AMδ
8	$\mid\ [\mathrm{P}]A_0$	2-7 [P]I

1	$\mid\ \langle\mathrm{F}\rangle A_0$	P
2	$\mid\ \langle\mathrm{F}\rangle B_0$	P
3	$\mid\ [\mathrm{F}](A \supset [\mathrm{F}]A)_0$	P
4	$\mid\ \underline{[\mathrm{F}](B \supset [\mathrm{F}]B)_0}$	P
5	$\mid\ \mid\ 0.1$	A (g, 1$\langle$F$\rangle$E)
6	$\mid\ \mid\ \underline{A_1}$	
7	$\mid\ \mid\ (A \supset [\mathrm{F}]A)_1$	3,5 [F]E
8	$\mid\ \mid\ [\mathrm{F}]A_1$	7,6 $\supset$E
9	$\mid\ \mid\ \mid\ 0.2$	A (g, 2$\langle$F$\rangle$E)
10	$\mid\ \mid\ \mid\ \underline{B_2}$	
11	$\mid\ \mid\ \mid\ (B \supset [\mathrm{F}]B)_2$	2,9 [F]E
12	$\mid\ \mid\ \mid\ [\mathrm{F}]B_2$	11,10 $\supset$E
13	$\mid\ \mid\ \mid\ \mid\ \underline{1.2}$	A (g, 5,9 AMφ)
14	$\mid\ \mid\ \mid\ \mid\ A_2$	8,13 [F]E
15	$\mid\ \mid\ \mid\ \mid\ (A \wedge B)_2$	10,14 $\wedge$I
16	$\mid\ \mid\ \mid\ \mid\ \langle\mathrm{F}\rangle(A \wedge B)_0$	9,15 $\langle$F$\rangle$I
17	$\mid\ \mid\ \mid\ \mid\ \underline{1 = 2}$	A (g, 5,9 AMφ)
18	$\mid\ \mid\ \mid\ \mid\ A_2$	6,17 =E
19	$\mid\ \mid\ \mid\ \mid\ (A \wedge B)_2$	18,10 $\wedge$I
20	$\mid\ \mid\ \mid\ \mid\ \langle\mathrm{F}\rangle(A \wedge B)_0$	9,19 $\langle$F$\rangle$I
21	$\mid\ \mid\ \mid\ \mid\ \underline{2.1}$	A (g, 5,9 AMφ)
22	$\mid\ \mid\ \mid\ \mid\ B_1$	12,21 [F]E
23	$\mid\ \mid\ \mid\ \mid\ (A \wedge B)_1$	6,22 $\wedge$I
24	$\mid\ \mid\ \mid\ \mid\ \langle\mathrm{F}\rangle(A \wedge B)_0$	5,23 $\langle$F$\rangle$I
25	$\mid\ \mid\ \mid\ \langle\mathrm{F}\rangle(A \wedge B)_0$	5,9,13-16,17-20,21-24 AMφ
26	$\mid\ \mid\ \langle\mathrm{F}\rangle(A \wedge B)_0$	2,9-25 $\langle$F$\rangle$E
27	$\mid\ \langle\mathrm{F}\rangle(A \wedge B)_0$	1,5-26 $\langle$F$\rangle$E

2.3 Soundness and Completeness

Preliminaries (excluding *NKv*): Begin with generalized notions of validity. For a model $\langle W, R, v\rangle$, let m be a map from subscripts into W. Say $\langle W, R, v\rangle_m$ is $\langle W, R, v\rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$, and for each $s = t \in \Gamma$, $m(s) = m(t)$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VK$_\alpha^{(t)*}$ $\Gamma \models^*_{K_\alpha^{(t)}} A_s$ $[\![s.t \,/\, s = t]\!]$ iff there is no $K_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R \,/\, m(s) \neq m(t)]\!]$.

NK$_\alpha^{(t)*}$ $\Gamma \vdash^*_{NK_\alpha^{(t)}} A_s$ $[\![s.t \,/\, s = t]\!]$ iff there is an $NK_\alpha^{(t)}$ derivation of A_s $[\![s.t \,/\, s = t]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $s.t$ or $s = t$). This is obvious for NK$_\alpha^{(t)*}$. In the other case, there is a $\langle W, R, v\rangle_m$ that makes all the members of Γ_0 true and A_0 false just in case there *is* a world in $\langle W, R, v\rangle$ that makes the unsubscripted members of Γ true and A false. For the following, cases omitted are like ones worked, and so left to the reader.

Theorem 2.1 *$NK_\alpha^{(t)}$ is sound: If $\Gamma \vdash_{NK_\alpha^{(t)}} A$ then $\Gamma \models_{K_\alpha^{(t)}} A$.*

L2.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{K_\alpha^{(t)}} P_s$ $[\![s.t \,/\, s = t]\!]$, then $\Gamma' \models^*_{K_\alpha^{(t)}} P_s$ $[\![s.t \,/\, s = t]\!]$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{K_\alpha^{(t)}} P_s$ $[\![s.t \,/\, s = t]\!]$, but $\Gamma' \not\models^*_{K_\alpha^{(t)}} P_s$ $[\![s.t \,/\, s = t]\!]$. From the latter, by VK$_\alpha^{(t)*}$, there is some $K_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma') = 1$ but $v_{m(s)}(P) = 0$ $[\![\langle m(s), m(t)\rangle \notin R \,/\, m(s) \neq m(t)]\!]$. But since $v_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $v_m(\Gamma) = 1$; so $v_m(\Gamma) = 1$ but $v_{m(s)}(P) = 0$ $[\![\langle m(s), m(t)\rangle \notin R \,/\, m(s) \neq m(t)]\!]$; so by VK$_\alpha^{(t)*}$, $\Gamma \not\models^*_{K_\alpha^{(t)}} P_s$ $[\![s.t \,/\, s = t]\!]$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{K_\alpha^{(t)}} P_s$ $[\![s.t \,/\, s = t]\!]$, then $\Gamma' \models^*_{K_\alpha^{(t)}} P_s$ $[\![s.t \,/\, s = t]\!]$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NK^{(t)}_\alpha} \mathcal{P}$ then $\Gamma \models^*_{K^{(t)}_\alpha} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are formulas with subscript 0. Suppose $\Gamma \vdash^*_{NK^{(t)}_\alpha} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{K^{(t)}_\alpha} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $\llbracket s.t \,/\, s = t \rrbracket$. Then $\Gamma_1 = \{A_s\}$ $\llbracket \{s.t\} \,/\, \{s = t\} \rrbracket$; so for any $\langle W, R, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $\llbracket \langle m(s), m(t)\rangle \in R \,/\, m(s) = m(t) \rrbracket$; so there is no $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $\llbracket \langle m(s), m(t)\rangle \notin R \,/\, m(s) \neq m(t) \rrbracket$. So by $\mathrm{VK}^{(t)*}_\alpha$, $\Gamma_1 \models^*_{K^{(t)}_\alpha} A_s$ $\llbracket s.t \,/\, s = t \rrbracket$, where this is just to say, $\Gamma_1 \models^*_{K^{(t)}_\alpha} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^*_{K^{(t)}_\alpha} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I, $\equiv$E, or, depending on the system,$\Box$I, $[\mathrm{F}]$I, $[\mathrm{P}]$I, $\Box$E, $[\mathrm{F}]$E, $[\mathrm{P}]$E, $\Diamond$I, $\langle\mathrm{F}\rangle$I, $\langle\mathrm{P}\rangle$I, $\Diamond$E, $\langle\mathrm{F}\rangle$E, $\langle\mathrm{P}\rangle$E, AMη, AMη', AMρ, AMσ, AMτ, AMδ, =E, AMφ, or AMβ. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\supset$I)

($\supset$E) If $\mathcal{P}_k$ arises by $\supset$E, then the picture is like this,

$$\begin{array}{r|l} i & (A \supset B)_s \\ j & A_s \\ & \\ k & B_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is B_s. By assumption, $\Gamma_i \models^*_{K^{(t)}_\alpha} (A \supset B)_s$ and $\Gamma_j \models^*_{K^{(t)}_\alpha} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K^{(t)}_\alpha} (A \supset B)_s$ and $\Gamma_k \models^*_{K^{(t)}_\alpha} A_s$. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} B_s$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is some $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(s)}(A \supset B) = 1$ and $v_{m(s)}(A) = 1$; from the former, by TK($\supset$), $v_{m(s)}(A) = 0$ or $v_{m(s)}(B) = 1$; so $v_{m(s)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} B_s$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

($\wedge$I)

($\wedge$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & A_s \\ \cline{3-3}
 & & \\
 & & \\
i & & B_t \\
j & & \neg B_t \\
k & \multicolumn{2}{l}{\neg A_s}
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{K^{(t)}_\alpha} B_t$ and $\Gamma_j \models^*_{K^{(t)}_\alpha} \neg B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L2.1, $\Gamma_k \cup \{A_s\} \models^*_{K^{(t)}_\alpha} B_t$ and $\Gamma_k \cup \{A_s\} \models^*_{K^{(t)}_\alpha} \neg B_t$. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} \neg A_s$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TK($\neg$), $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(t)}(B) = 1$ and $v_{m(t)}(\neg B) = 1$; from the latter, by TK($\neg$), $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} \neg A_s$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

($\neg$E)

($\vee$I)

($\vee$E) If $\mathcal{P}_k$ arises by $\vee$E, then the picture is like this,

$$\begin{array}{r|l}
h & (A \vee B)_s \\
 & \quad | \; \underline{A_s} \\
 & \quad | \\
i & \quad | \; C_t \\
 & \quad | \; \underline{B_s} \\
 & \quad | \\
j & \quad | \; C_t \\
k & C_t
\end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is C_t. By assumption, $\Gamma_h \models^*_{K^{(t)}_\alpha} (A \vee B)_s$, $\Gamma_i \models^*_{K^{(t)}_\alpha} C_t$ and $\Gamma_j \models^*_{K^{(t)}_\alpha} C_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{B_s\}$; so by L2.1, $\Gamma_k \models^*_{K^{(t)}_\alpha} (A \vee B)_s$, $\Gamma_k \cup \{A_s\} \models^*_{K^{(t)}_\alpha} C_t$ and $\Gamma_k \cup \{B_s\} \models^*_{K^{(t)}_\alpha} C_t$. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} C_t$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is some $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(C) = 0$. Since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(s)}(A \vee B) = 1$; so by TK($\vee$), $v_{m(s)}(A) = 1$ or $v_{m(s)}(B) = 1$. Suppose, for the moment, that $v_{m(s)}(A) = 1$; then $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(t)}(C) = 1$; this is impossible; reject the assumption: $v_{m(s)}(A) \neq 1$; so $v_{m(s)}(B) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(B) = 1$; so $v_m(\Gamma_k \cup \{B_s\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(t)}(C) = 1$; this is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} C_t$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

($\equiv$I)

($\equiv$E)

($\Box$I) If $\mathcal{P}_k$ arises by $\Box$I, then the picture is like this,

$$\begin{array}{r|l}
 & \quad | \; \underline{s.t} \\
 & \quad | \\
j & \quad | \; A_t \\
k & \Box A_s
\end{array}$$

where $j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\Box A_s$. By assumption, $\Gamma_j \models^*_{K^{(t)}_\alpha} A_t$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{s.t\}$; so by L2.1, $\Gamma_k \cup \{s.t\} \models^*_{K^{(t)}_\alpha} A_t$. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} \Box A_s$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but

$v_{m(s)}(\Box A) = 0$; so by TK($\Box$), there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by $\mathrm{VK}_\alpha^{(t)*}$, $v_{m'(t)}(A) = 1$. But $m'(t) = w$; so $v_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_\alpha^{(t)}} \Box A_s$, which is to say, $\Gamma_k \models^*_{K_\alpha^{(t)}} \mathcal{P}_k$.

([F]I)

([P]I) If $\mathcal{P}_k$ arises by [P]I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & s.t \\ \cline{3-3}
 & & \\
j & & A_s \\
k & \multicolumn{2}{l}{[\mathrm{P}]A_t}
\end{array}$$

where $j < k$, s does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $[\mathrm{P}]A_t$. By assumption, $\Gamma_j \models^*_{K_\alpha^{(t)}} A_s$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{s.t\}$; so by L2.1, $\Gamma_k \cup \{s.t\} \models^*_{K_\alpha^{(t)}} A_s$. Suppose $\Gamma_k \not\models^*_{K_\alpha^{(t)}} [\mathrm{P}]A_t$; then by $\mathrm{VK}_\alpha^{(t)*}$, there is a $K_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}([\mathrm{P}]A) = 0$; so by TK([P]), there is some $w \in W$ such that $wRm(t)$ and $v_w(A) = 0$. Now consider a map m' like m except that $m'(s) = w$, and consider $\langle W, R, v\rangle_{m'}$; since s does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = w$ and $m'(t) = m(t)$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by $\mathrm{VK}_\alpha^{(t)*}$, $v_{m'(s)}(A) = 1$. But $m'(s) = w$; so $v_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_\alpha^{(t)}} [\mathrm{P}]A_t$, which is to say, $\Gamma_k \models^*_{K_\alpha^{(t)}} \mathcal{P}_k$.

($\Box$E) If $\mathcal{P}_k$ arises by $\Box$E, then the picture is like this,

$$\begin{array}{r|l}
i & \Box A_s \\
j & s.t \\
 & \\
k & A_t
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{K_\alpha^{(t)}} \Box A_s$ and $\Gamma_j \models^*_{K_\alpha^{(t)}} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K_\alpha^{(t)}} \Box A_s$ and $\Gamma_k \models^*_{K_\alpha^{(t)}} s.t$. Suppose $\Gamma_k \not\models^*_{K_\alpha^{(t)}} A_t$;

then by VK$_{\alpha}^{(t)*}$, there is some $K_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VK$_{\alpha}^{(t)*}$, $v_{m(s)}(\Box A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, by TK($\Box$), any w such that $m(s)Rw$ has $v_w(A) = 1$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_\alpha^{(t)}} A_t$, which is to say, $\Gamma_k \models^*_{K_\alpha^{(t)}} \mathcal{P}_k$.

([F]E)

([P]E) If $\mathcal{P}_k$ arises by [P]E, then the picture is like this,

$$\begin{array}{r|l}
i & [\mathrm{P}]A_t \\
j & s.t \\
 & \\
k & A_s
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is A_s. By assumption, $\Gamma_i \models^*_{K_\alpha^{(t)}} [\mathrm{P}]A_t$ and $\Gamma_j \models^*_{K_\alpha^{(t)}} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K_\alpha^{(t)}} [\mathrm{P}]A_t$ and $\Gamma_k \models^*_{K_\alpha^{(t)}} s.t$. Suppose $\Gamma_k \not\models^*_{K_\alpha^{(t)}} A_s$; then by VK$_{\alpha}^{(t)*}$, there is some $K_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VK$_{\alpha}^{(t)*}$, $v_{m(t)}([\mathrm{P}]A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, by TK([P]), any w such that $wRm(t)$ has $v_w(A) = 1$; so $v_{m(s)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_\alpha^{(t)}} A_s$, which is to say, $\Gamma_k \models^*_{K_\alpha^{(t)}} \mathcal{P}_k$.

($\Diamond$I)

($\langle$F$\rangle$I)

($\langle$P$\rangle$I)

($\Diamond$E) If $\mathcal{P}_k$ arises by $\Diamond$E, then the picture is like this,

$$\begin{array}{r|l|l}
i & \multicolumn{2}{l}{\Diamond A_s} \\
 & & A_t \\
 & & s.t \\ \cline{3-3}
 & & \\
j & & B_u \\
k & \multicolumn{2}{l}{B_u}
\end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \vDash^*_{K^{(t)}_\alpha} \Diamond A_s$ and $\Gamma_j \vDash^*_{K^{(t)}_\alpha} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t, s.t\}$; so by L2.1, $\Gamma_k \vDash^*_{K^{(t)}_\alpha} \Diamond A_s$ and $\Gamma_k \cup \{A_t, s.t\} \vDash^*_{K^{(t)}_\alpha} B_u$. Suppose $\Gamma_k \nvDash^*_{K^{(t)}_\alpha} B_u$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(s)}(\Diamond A) = 1$; so by $\mathrm{TK}(\Diamond)$, there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $v_{m'(t)}(A) = 1$ and $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{A_t, s.t\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{K^{(t)}_\alpha} B_u$, which is to say, $\Gamma_k \vDash^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

($\langle\text{F}\rangle$E)

($\langle\text{P}\rangle$E) If $\mathcal{P}_k$ arises by $\langle\text{P}\rangle$E, then the picture is like this,

$$
\begin{array}{r|l}
i & \langle \text{P}\rangle A_t \\
 & \;\vline\; s.t \\
 & \;\vline\; \underline{A_s} \\
 & \;\vline\; \\
 & \;\vline\; \\
j & \;\vline\; B_u \\
k & B_u
\end{array}
$$

where $i, j < k$, s does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \vDash^*_{K^{(t)}_\alpha} \langle\text{P}\rangle A_t$ and $\Gamma_j \vDash^*_{K^{(t)}_\alpha} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s.t, A_s\}$; so by L2.1, $\Gamma_k \vDash^*_{K^{(t)}_\alpha} \langle\text{P}\rangle A_t$ and $\Gamma_k \cup \{s.t, A_s\} \vDash^*_{K^{(t)}_\alpha} B_u$. Suppose $\Gamma_k \nvDash^*_{K^{(t)}_\alpha} B_u$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(t)}(\langle\text{P}\rangle A) = 1$; so by $\mathrm{TK}(\langle\text{P}\rangle)$, there is some $w \in W$ such that $wRm(t)$ and $v_w(A) = 1$. Now consider a map m' like m except that $m'(s) = w$, and consider $\langle W, R, v\rangle_{m'}$; since s does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = m(t)$ and $m'(s) = w$, $v_{m'(s)}(A) = 1$ and $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t, A_s\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m'(u)}(B) = 1$. But since

$s \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} B_u$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

(AMη) If $\mathcal{P}_k$ arises by AMη, then the picture is like this,

$$\begin{array}{r|l|l}
 & & s.t \\ \cline{3-3}
 & & \\
j & & A_u \\
k & \multicolumn{2}{l}{A_u}
\end{array}$$

where $j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is A_u. Where this rule is included in $NK^{(t)}_\alpha$, $K^{(t)}_\alpha$ includes condition η. By assumption, $\Gamma_j \models^*_{K^{(t)}_\alpha} A_u$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{s.t\}$; so by L2.1, $\Gamma_k \cup \{s.t\} \models^*_{K^{(t)}_\alpha} A_u$. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} A_u$; then by $\text{VK}^{(t)*}_\alpha$, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(A) = 0$. By condition η, there is a $w \in W$ such that $m(s)Rw$; consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by $\text{VK}^{(t)*}_\alpha$, $v_{m'(u)}(A) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} A_u$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

(AMη')

(AMρ) If $\mathcal{P}_k$ arises by AMρ, then the picture is like this,

$$\begin{array}{r|l}
 & \\
k & s.s
\end{array}$$

where $\mathcal{P}_k$ is $s.s$. Where this rule is in $NK^{(t)}_\alpha$, $K^{(t)}_\alpha$ includes condition ρ. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} s.s$; then by $\text{VK}^{(t)*}_\alpha$, there is some $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(s)\rangle \notin R$. But by condition ρ, for any $x \in W$, $\langle x, x\rangle \in R$; so $\langle m(s), m(s)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} s.s$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

(AMσ) If $\mathcal{P}_k$ arises by AMσ, then the picture is like this,

$$\begin{array}{r|l} j & s.t \\ & \\ k & t.s \end{array}$$

where $j < k$ and $\mathcal{P}_k$ is $t.s$. Where this rule is in $NK_\alpha^{(t)}$, $K_\alpha^{(t)}$ includes condition σ. By assumption, $\Gamma_j \models^*_{K_\alpha^{(t)}} s.t$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K_\alpha^{(t)}} s.t$. Suppose $\Gamma_k \not\models^*_{K_\alpha^{(t)}} t.s$; then by $\mathrm{VK}_\alpha^{(t)*}$, there is some $K_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(t), m(s)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}_\alpha^{(t)*}$, $\langle m(s), m(t)\rangle \in R$; and by condition σ, for any $\langle x, y\rangle \in R$, $\langle y, x\rangle \in R$; so $\langle m(t), m(s)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_\alpha^{(t)}} t.s$, which is to say, $\Gamma_k \models^*_{K_\alpha^{(t)}} \mathcal{P}_k$.

(AMτ) If $\mathcal{P}_k$ arises by AMτ, then the picture is like this,

$$\begin{array}{r|l} i & s.t \\ j & t.u \\ & \\ k & s.u \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $s.u$. Where this rule is in $NK_\alpha^{(t)}$, $K_\alpha^{(t)}$ includes condition τ. By assumption, $\Gamma_i \models^*_{K_\alpha^{(t)}} s.t$ and $\Gamma_j \models^*_{K_\alpha^{(t)}} t.u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K_\alpha^{(t)}} s.t$ and $\Gamma_k \models^*_{K_\alpha^{(t)}} t.u$. Suppose $\Gamma_k \not\models^*_{K_\alpha^{(t)}} s.u$; then by $\mathrm{VK}_\alpha^{(t)*}$, there is some $K_\alpha^{(t)}$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(u)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}_\alpha^{(t)*}$, $\langle m(s), m(t)\rangle \in R$ and $\langle m(t), m(u)\rangle \in R$; and by condition τ, for any $\langle x, y\rangle, \langle y, z\rangle \in R$, $\langle x, z\rangle \in R$; so $\langle m(s), m(u)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_\alpha^{(t)}} s.u$, which is to say, $\Gamma_k \models^*_{K_\alpha^{(t)}} \mathcal{P}_k$.

(AMδ) If $\mathcal{P}_k$ arises by AMδ, then the picture is like this,

$$\begin{array}{r|l} i & s.t \\ & \mid s.a \\ & \mid \underline{a.t} \\ & \mid \\ j & \mid A_u \\ k & A_u \end{array}$$

where $i, j < k$, a does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is A_u.

Where this rule is included in $NK^{(t)}_\alpha$, $K^{(t)}_\alpha$ includes condition δ. By assumption, $\Gamma_i \models^*_{K^{(t)}_\alpha} s.t$ and $\Gamma_j \models^*_{K^{(t)}_\alpha} A_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s.a, a.t\}$; so by L2.1, $\Gamma_k \models^*_{K^{(t)}_\alpha} s.t$ and $\Gamma_k \cup \{s.a, a.t\} \models^*_{K^{(t)}_\alpha} A_u$. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} A_u$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}^{(t)*}_\alpha$, $\langle m(s), m(t)\rangle \in R$; and by condition δ, if $\langle x, y\rangle \in R$ then for some z, $\langle x, z\rangle \in R$ and $\langle z, y\rangle \in R$; so there is a $w \in W$ such that $m(s)Rw$ and $wRm(t)$; consider a map m' like m except that $m'(a) = w$, and consider $\langle W, R, v\rangle_{m'}$; since a does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$, $m'(a) = w$, and $m'(t) = m(t)$, $\langle m'(s), m'(a)\rangle \in R$ and $\langle m'(a), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.a, a.t\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m'(u)}(A) = 1$. But since $a \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} A_u$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

(=E) If P_k arises by =E, then the picture is like this,

$$\begin{array}{r|l} i & s = t \\ j & \mathcal{A}(s) \\ & \\ k & \mathcal{A}(t) \end{array} \qquad \text{or} \qquad \begin{array}{r|l} i & t = s \\ j & \mathcal{A}(s) \\ & \\ k & \mathcal{A}(t) \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\mathcal{A}(t)$ in both cases. By assumption, $\Gamma_i \models^*_{K^{(t)}_\alpha}$ $s = t$ / $t = s$ and $\Gamma_j \models^*_{K^{(t)}_\alpha} \mathcal{A}(s)$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L2.1, $\Gamma_k \models^*_{K^{(t)}_\alpha} s = t$ / $t = s$ and $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{A}(s)$. In both cases, $\mathcal{A}(s)$ is of the sort, A_u, $u = v$ or $u.v$ where one u or v is s. Suppose $\mathcal{A}(s)$ is A_s and $\Gamma_k \not\models^*_{K^{(t)}_\alpha} A_t$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is some $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}^{(t)*}_\alpha$, $m(s) = m(t)$ / $m(t) = m(s)$ and $v_{m(s)}(A) = 1$; but since $m(s) = m(t)$ / $m(t) = m(s)$, $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} A_{(t)}$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$. And similarly in the other cases.

(AMφ) If $\mathcal{P}_k$ arises by AMφ, then the picture is like this,

$$
\begin{array}{r|c|l}
f & \multicolumn{2}{l}{r.s} \\
g & \multicolumn{2}{l}{r.t} \\
 & & s.t \\ \cline{3-3}
 & & \\
h & & A_u \\
 & & s = t \\ \cline{3-3}
 & & \\
i & & A_u \\
 & & t.s \\ \cline{3-3}
 & & \\
j & & A_u \\
k & \multicolumn{2}{l}{A_u}
\end{array}
$$

where $f, g, h, i, j < k$ and $\mathcal{P}_k$ is A_u. Where this rule is included in $NK^{(t)}_\alpha$, $K^{(t)}_\alpha$ includes condition φ. By assumption, $\Gamma_f \models^*_{K^{(t)}_\alpha} r.s$, $\Gamma_g \models^*_{K^{(t)}_\alpha} r.t$, $\Gamma_h \models^*_{K^{(t)}_\alpha} A_u$, $\Gamma_i \models^*_{K^{(t)}_\alpha} A_u$, and $\Gamma_j \models^*_{K^{(t)}_\alpha} A_u$; but by the nature of access, $\Gamma_f \subseteq \Gamma_k$, $\Gamma_g \subseteq \Gamma_k$, $\Gamma_h \subseteq \Gamma_k \cup \{s.t\}$, $\Gamma_i \subseteq \Gamma_k \cup \{s = t\}$, and $\Gamma_j \subseteq \Gamma_k \cup \{t.s\}$; so by L2.1, $\Gamma_k \models^*_{K^{(t)}_\alpha} r.s$, $\Gamma_k \models^*_{K^{(t)}_\alpha} r.t$, $\Gamma_k \cup \{s.t\} \models^*_{K^{(t)}_\alpha} A_u$, $\Gamma_k \cup \{s = t\} \models^*_{K^{(t)}_\alpha} A_u$, and $\Gamma_k \cup \{t.s\} \models^*_{K^{(t)}_\alpha} A_u$. Suppose $\Gamma_k \not\models^*_{K^{(t)}_\alpha} A_u$; then by $\mathrm{VK}^{(t)*}_\alpha$, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VK}^{(t)*}_\alpha$, $\langle m(r), m(s)\rangle \in R$ and $\langle m(r), m(t)\rangle \in R$; and by condition φ, if $\langle x, y\rangle \in R$ and $\langle x, z\rangle \in R$ then either $\langle y, z\rangle \in R$, $y = z$, or $\langle z, y\rangle \in R$; so either (i) $m(s)Rm(t)$, (ii) $m(s) = m(t)$, or (iii) $m(t)Rm(s)$. Suppose (i) $m(s)Rm(t)$; then $v_m(\Gamma_k \cup \{s.t\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(u)}(A) = 1$. Suppose (ii) $m(s) = m(t)$; then $v_m(\Gamma_k \cup \{s = t\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(u)}(A) = 1$. Suppose (iii) $m(t)Rm(s)$; then $v_m(\Gamma_k \cup \{t.s\}) = 1$; so by $\mathrm{VK}^{(t)*}_\alpha$, $v_{m(u)}(A) = 1$. In any case, $v_{m(u)}(A) = 1$. This is impossible; reject the original assumption: $\Gamma_k \models^*_{K^{(t)}_\alpha} A_u$, which is to say, $\Gamma_k \models^*_{K^{(t)}_\alpha} \mathcal{P}_k$.

(AMβ)

For any i, $\Gamma_i \models^*_{K^{(t)}_\alpha} \mathcal{P}_i$.

The argument for NKv is similar (simpler) and so omitted.

Theorem 2.2 *$NK^{(t)}_\alpha$ is complete: if $\Gamma \vDash_{K^{(t)}_\alpha} A$ then $\Gamma \vdash_{NK^{(t)}_\alpha} A$.*

Suppose $\Gamma \vDash_{K^{(t)}_\alpha} A$; then $\Gamma_0 \vDash^*_{K^{(t)}_\alpha} A_0$; we show that $\Gamma_0 \vdash^*_{NK^{(t)}_\alpha} A_0$. Again, this reduces to the standard notion. The method of our proof has advantages (especially for the quantificational case) over standard approaches to completeness for modal logic. Roughly, we construct a single set which is maximal and consistent relative to *subscripted* formulas, and use this to specify the model. The resultant proof is thus kept structurally parallel to the classical case. For the following, fix on some particular constraint(s) α. Then definitions of *consistency* etc. are relative to it.

Con Γ is consistent iff there is no A_s such that $\Gamma \vdash^*_{NK^{(t)}_\alpha} A_s$ and $\Gamma \vdash^*_{NK^{(t)}_\alpha} \neg A_s$.

L2.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NK^{(t)}_\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^*_{NK^{(t)}_\alpha} \neg P_s$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^*_{NK^{(t)}_\alpha} A_t$ and $\Gamma \cup \{P_s\} \vdash^*_{NK^{(t)}_\alpha} \neg A_t$. But then we can argue,

1	Γ	
2	$\quad P_s$	A (c, $\neg$I)
3	$\quad A_t$	from $\Gamma \cup \{P_s\}$
4	$\quad \neg A_t$	from $\Gamma \cup \{P_s\}$
5	$\neg P_s$	2-4 $\neg$I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^*_{NK^{(t)}_\alpha} \neg P_s$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NK^{(t)}_\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

L2.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1$ $\mathcal{P}_2 \ldots$

Proof by construction: Order non-subscripted formulas A, B, $C \ldots$ in the usual way. Then form a grid with formulas A, B, $C \ldots$ ordered across the top, and subscripts 1, 2, 3 ... down the side.

$$\begin{tikzcd}
A_1 \arrow[r] & B_1 \arrow[dl] & C_1 & \ldots \\
A_2 \arrow[d] & B_2 \arrow[ur] & C_2 & \\
A_3 \arrow[ur] & B_3 & C_3 & \\
\vdots & & &
\end{tikzcd}$$

Order the members of the resultant grid, A_1, B_1, $A_2 \dots$ moving along the arrows from the upper left corner, down and to the right.[3]

In addition, there is an enumeration of these formulas with access relations $s.t$, with pairs of the sort $s.t$ / $u.v$, and with expressions of the sort $s = t$.

Proof by construction.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NK^{(t)}_\alpha} A_s$ or $\Gamma \vdash^*_{NK^{(t)}_\alpha} \neg A_s$.

SGT Γ is a SCAPEGOAT set iff for every formula of the form $\neg\Box A_s$, if $\Gamma \vdash^*_{NK^{(t)}_\alpha} \neg\Box A_s$ then there is some t such that $\Gamma \vdash^*_{NK^{(t)}_\alpha} s.t$ and $\Gamma \vdash^*_{NK^{(t)}_\alpha} \neg A_t$; and similarly for every formula of the form $\neg[\mathrm{F}]A_s$; but for every formula of the form $\neg[\mathrm{P}]A_s$, if $\Gamma \vdash^*_{NK^{(t)}_\alpha} \neg[\mathrm{P}]A_s$ then there is some t such that $\Gamma \vdash^*_{NK^{(t)}_\alpha} t.s$ and $\Gamma \vdash^*_{NK^{(t)}_\alpha} \neg A_t$.

Γ is a SCAPEGOAT set for AMδ iff for any access relation $s.t$, if $\Gamma \vdash^*_{NK^{(t)}_\alpha} s.t$, then there is a u such that $\Gamma \vdash^*_{NK^{(t)}_\alpha} s.u$ and $\Gamma \vdash^*_{NK^{(t)}_\alpha} u.t$.

Γ is a SCAPEGOAT set for AMφ iff for any access relation $r.s$ / $r.t$, if $\Gamma \vdash^*_{NK^{(t)}_\alpha} r.s$ and $\Gamma \vdash^*_{NK^{(t)}_\alpha} r.t$, then $\Gamma \vdash^*_{NK^{(t)}_\alpha} s.t$, $\Gamma \vdash^*_{NK^{(t)}_\alpha} s = t$, or $\Gamma \vdash^*_{NK^{(t)}_\alpha} t.s$.

Similarly for AMβ.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L2.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \dots$ of all the subscripted formulas, together with all the access relations $s.t$ if δ is in $K^{(t)}_\alpha$, along with pairs $s.t$ / $u.v$ and expressions $s = t$ if φ or β are in $K^{(t)}_\alpha$; let $\mathcal{E}_0$ be this enumeration. Then for the first expression $\mathcal{P}$ in $\mathcal{E}_{i-1}$ such that all its subscripts are 0 or introduced in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $\mathcal{P}$, and set,

---

[3] As for rational numbers; see, e.g., [6, §2.1.1].

(i) $\Omega_i = \Omega_{i-1}$ if $\Omega_{i-1} \cup \{\mathcal{P}\}$ is inconsistent

$\Omega_{i^*} = \Omega_{i-1} \cup \{\mathcal{P}\}$ if $\Omega_{i-1} \cup \{\mathcal{P}\}$ is consistent

and

(ii) $\Omega_i = \Omega_{i^*}$ if $\mathcal{P}$ is not of the form $\neg\Box A_s$, $\neg[\mathrm{F}]A_s$, $\neg[\mathrm{P}]A_s$, $s.t$, $r.s$ / $r.t$, or $s.r$ / $t.r$

(iii) $\Omega_i = \Omega_{i^*} \cup \{s.u, \neg P_u\}$ if $\mathcal{P}$ is of the form $\neg\Box A_s$ or $\neg[\mathrm{F}]A_s$

(iv) $\Omega_i = \Omega_{i^*} \cup \{u.s, \neg P_u\}$ if $\mathcal{P}$ is of the form $\neg[\mathrm{P}]A_s$

(v) $\Omega_i = \Omega_{i^*} \cup \{s.u, u.t\}$ if $\mathcal{P}$ is of the form $s.t$

(vi) $\Omega_i = \Omega_{i^*} \cup \{s.t\}$ if $\mathcal{P}$ is of the form $r.s$ / $r.t$ or $s.r$ / $t.r$ and $\Omega_{i^*} \cup \{s.t\}$ is consistent;

otherwise

$\Omega_i = \Omega_{i^*} \cup \{s = t\}$ if $\mathcal{P}$ is of the form $r.s$ / $r.t$ or $s.r$ / $t.r$ and $\Omega_{i^*} \cup \{s = t\}$ is consistent;

otherwise

$\Omega_i = \Omega_{i^*} \cup \{t.s\}$ if $\mathcal{P}$ is of the form $r.s$ / $r.t$ or $s.r$ / $t.r$

-where u is the first subscript not introduced in Ω_{i^*}

then

$\Gamma' = \bigcup_{i \geq 0} \Omega_i$

Note that there is always sure to be a subscript u not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L2.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} A_s$ and $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} \neg A_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NK^{(t)}_\alpha} \neg A_s$ then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg A_s$; but $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} \neg A_s$; so $\Omega_{i-1} \nvdash^*_{NK^{(t)}_\alpha} \neg A_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L2.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$, (iii) $\Omega_{k^*} \cup \{s.u, \neg P_u\}$, (iv) $\Omega_{k^*} \cup \{u.s, \neg P_u\}$, (v) $\Omega_{k^*} \cup \{s.u, u.t\}$, (vi) $\Omega_{k^*} \cup \{s.t\}$ or $\Omega_{k^*} \cup \{s = t\}$ or $\Omega_{k^*} \cup \{t.s\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$. Then by construction, $\Omega_{k-1} \cup \{\mathcal{P}\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.u, \neg P_u\}$. In this case, as above, Ω_{k^*} is consistent and by construction, $\neg\Box P_s \in \Omega_{k^*}$ or $\neg[\text{F}]P_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_v and $\neg A_v$ such that $\Omega_{k^*} \cup \{s.u, \neg P_u\} \vdash^*_{NK^{(t)}_\alpha} A_v$ and $\Omega_{k^*} \cup \{s.u, \neg P_u\} \vdash^*_{NK^{(t)}_\alpha} \neg A_v$. So, for the first case, reason as follows,

$$
\begin{array}{r|l l}
1 & \Omega_{k^*} & \\
2 & \mid\ \underline{s.u} & \text{A } (g, \Box\text{I}) \\
3 & \mid\ \mid\ \underline{\neg P_u} & \text{A } (c, \neg\text{E}) \\
4 & \mid\ \mid\ A_v & \text{from } \Omega_{k^*} \cup \{s.u, \neg P_u\} \\
5 & \mid\ \mid\ \neg A_v & \text{from } \Omega_{k^*} \cup \{s.u, \neg P_u\} \\
6 & \mid\ P_u & \text{3-5 } \neg\text{E} \\
7 & \Box P_s & \text{2-6 } \Box\text{I}
\end{array}
$$

where, by construction, u is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NK^{(t)}_\alpha} \Box P_s$; but $\neg\Box P_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NK^{(t)}_\alpha} \neg\Box P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent. And similarly if $\neg[\text{F}]P_s \in \Omega_{k^*}$.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{u.s, \neg P_u\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg[\text{P}]P_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_v and $\neg A_v$ such that $\Omega_{k^*} \cup \{u.s, \neg P_u\} \vdash^*_{NK^{(t)}_\alpha} A_v$ and $\Omega_{k^*} \cup \{u.s, \neg P_u\} \vdash^*_{NK^{(t)}_\alpha} \neg A_v$. So reason as follows,

$$\begin{array}{r|lll}
1 & \Omega_{k^*} & & \\
2 & \mid\; \underline{u.s} & & \text{A } (g,\ [\text{P}]\text{I}) \\
3 & \mid\; \mid\; \underline{\neg P_u} & & \text{A } (c,\ \neg\text{E}) \\
4 & \mid\; \mid\; A_v & & \text{from } \Omega_{k^*} \cup \{u.s, \neg P_u\} \\
5 & \mid\; \mid\; \neg A_v & & \text{from } \Omega_{k^*} \cup \{u.s, \neg P_u\} \\
6 & \mid\; P_u & & \text{3-5 } \neg\text{E} \\
7 & [\text{P}]P_s & & \text{2-6 } [\text{P}]\text{I}
\end{array}$$

where, by construction, u is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NK^{(t)}_\alpha} [\text{P}]P_s$; but $\neg[\text{P}]P_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NK^{(t)}_\alpha} \neg[\text{P}]P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(v) Suppose Ω_k is $\Omega_{k^*} \cup \{s.u, u.t\}$. In this case, as above, Ω_{k^*} is consistent and by construction, $s.t \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_v and $\neg A_v$ such that $\Omega_{k^*} \cup \{s.u, u.t\} \vdash^*_{NK^{(t)}_\alpha} A_v$ and $\Omega_{k^*} \cup \{s.u, u.t\} \vdash^*_{NK^{(t)}_\alpha} \neg A_v$. So reason as follows,

$$\begin{array}{r|lll}
1 & \Omega_{k^*} & & \\
2 & s.t & & \text{from } \Omega_{k^*} \\
3 & \mid\; s.u & & \text{A } (g,\ AM\delta) \\
4 & \mid\; \underline{u.t} & & \text{A } (g,\ AM\delta) \\
5 & \mid\; \mid\; \underline{\neg(A \wedge \neg A)_w} & & \text{A } (c,\ \neg\text{E}) \\
6 & \mid\; \mid\; A_v & & \text{from } \Omega_{k^*} \cup \{s.u, u.t\} \\
7 & \mid\; \mid\; \neg A_v & & \text{from } \Omega_{k^*} \cup \{s.u, u.t\} \\
8 & \mid\; (A \wedge \neg A)_w & & \text{5-7 } \neg\text{E} \\
9 & (A \wedge \neg A)_w & & \text{2, 3-8 } AM\delta \\
10 & A_w & & \text{9 } \wedge\text{E} \\
11 & \neg A_w & & \text{9 } \wedge\text{E}
\end{array}$$

where, by construction, u is not in Ω_{k^*} and u is not w. So $\Omega_{k^*} \vdash^*_{NK^{(t)}_\alpha} A_w$ and $\Omega_{k^*} \vdash^*_{NK^{(t)}_\alpha} \neg A_w$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(vi) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t\}$ or $\Omega_{k^*} \cup \{s = t\}$ or $\Omega_{k^*} \cup \{t.s\}$. In any case, as above, Ω_{k^*} is consistent and by construction, $r.s / r.t \in \Omega_{k^*}$ or $s.r$ / $t.r \in \Omega_{k^*}$. In the first case, by construction, $\Omega_{k^*} \cup \{s.t\}$ is consistent; so Ω_k is consistent. In the second case, by construction again, $\Omega_{k^*} \cup \{s = t\}$ is consistent; so Ω_k is consistent. If the third case, then $\Omega_{k^*} \cup \{s.t\}$ is inconsistent and $\Omega_{k^*} \cup \{s = t\}$ is inconsistent. Suppose Ω_k is inconsistent. Then there are A_u, $\neg A_u$, B_v, $\neg B_v$, C_w, and $\neg C_w$ such that $\Omega_{k^*} \cup \{s.t\} \vdash^*_{NK^{(t)}_\alpha} A_u$ and $\Omega_{k^*} \cup \{s.t\} \vdash^*_{NK^{(t)}_\alpha} \neg A_u$, $\Omega_{k^*} \cup \{s =$

$t\} \vdash^*_{NK^{(t)}_{\alpha}} B_v$ and $\Omega_{k^*} \cup \{s = t\} \vdash^*_{NK^{(t)}_{\alpha}} \neg B_v$, and $\Omega_{k^*} \cup \{t.s\} \vdash^*_{NK^{(t)}_{\alpha}} C_w$ and $\Omega_{k^*} \cup \{t.s\} \vdash^*_{NK^{(t)}_{\alpha}} \neg C_w$. So reason as follows,

1	Ω_{k^*}	
2	$r.s$	from Ω_{k^*}
3	$r.t$	from Ω_{k^*}
4	$\mid\ \underline{s.t}$	A (g, $AM\varphi$)
5	$\mid\ \mid\ \underline{\neg(D \wedge \neg D)_x}$	A (c, $\neg$E)
6	$\mid\ \mid\ A_u$	from $\Omega_{k^*} \cup \{s.t\}$
7	$\mid\ \mid\ \neg A_u$	from $\Omega_{k^*} \cup \{s.t\}$
8	$\mid\ (D \wedge \neg D)_x$	5-7 $\neg$E
9	$\mid\ \underline{s = t}$	A (g, $AM\varphi$)
10	$\mid\ \mid\ \underline{\neg(D \wedge \neg D)_x}$	A (c, $\neg$E)
11	$\mid\ \mid\ B_v$	from $\Omega_{k^*} \cup \{s = t\}$
12	$\mid\ \mid\ \neg B_v$	from $\Omega_{k^*} \cup \{s = t\}$
13	$\mid\ (D \wedge \neg D)_x$	10-12 $\neg$E
14	$\mid\ \underline{t.s}$	A (g, $AM\varphi$)
15	$\mid\ \mid\ \underline{\neg(D \wedge \neg D)_x}$	A (c, $\neg$E)
16	$\mid\ \mid\ C_w$	from $\Omega_{k^*} \cup \{t.s\}$
17	$\mid\ \mid\ \neg C_w$	from $\Omega_{k^*} \cup \{t.s\}$
18	$\mid\ (D \wedge \neg D)_x$	15-17 $\neg$E
19	$(D \wedge \neg D)_x$	2,3,4-8,9-13,14-18 $AM\varphi$
20	D_x	19 $\wedge$E
21	$\neg D_x$	19 $\wedge$E

So $\Omega_{k^*} \vdash^*_{NK^{(t)}_{\alpha}} D_x$ and $\Omega_{k^*} \vdash^*_{NK^{(t)}_{\alpha}} \neg D_x$. Similar reasoning follows for $s.r$ / $t.r \in \Omega_{k^*}$ with the rule $AM\beta$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L2.6 If Γ_0 is consistent, then Γ' is consistent.

Suppose Γ_0 is consistent, but Γ' is not; from the latter, there is some P_s such that $\Gamma' \vdash^*_{NK^{(t)}_{\alpha}} P_s$ and $\Gamma' \vdash^*_{NK^{(t)}_{\alpha}} \neg P_s$. Consider derivations D1 and D2 of these results, and the premises $\mathcal{P}_i \ldots \mathcal{P}_j$ of these derivations. By construction, there is an Ω_k with each of these premises as a member; so D1 and D2 are derivations from Ω_k; so Ω_k is not consistent. But since Γ_0 is consistent, by L2.5, Ω_k is consistent. This is impossible; reject the assumption: if Γ_0 is consistent then Γ' is consistent.

L2.7 If Γ_0 is consistent, then Γ' is a scapegoat set.

Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg\Box P_s$. By L2.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} \neg\neg\Box P_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg\Box P_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, \neg P_t\}$; so by construction, $s.t \in \Gamma'$ and $\neg P_t \in \Gamma'$; so $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg P_t$. (Similarly for [F] and [P].) So Γ' is a scapegoat set.

For AMδ. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$. By L2.6, Γ' is consistent; and by the constraints on subscripts, s and t are introduced in Γ'. Since $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$, Γ' has just the same consequences as $\Gamma' \cup \{s.t\}$; so $\Gamma' \cup \{s.t\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{s.t\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{s.t\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.u, u.t\}$; so by construction, $s.u, u.t \in \Gamma'$; so there is a u such that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.u$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} u.t$. So Γ' is a scapegoat set for AMδ.

For AMφ. Suppose Γ_0 is consistent, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} r.s$, and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} r.t$. By L2.6, Γ' is consistent; and by the constraints on subscripts, r, s, and t are introduced in Γ'. Since $\Gamma' \vdash^*_{NK^{(t)}_\alpha} r.s$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} r.t$, Γ' has just the same consequences as $\Gamma' \cup \{r.s \,/\, r.t\}$; so $\Gamma' \cup \{r.s \,/\, r.t\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{r.s \,/\, r.t\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{r.s \,/\, r.t\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t\}$, $\Omega_i = \Omega_{i^*} \cup \{s = t\}$, or $\Omega_i = \Omega_{i^*} \cup \{t.s\}$; so by construction, $s.t \in \Gamma'$, $s = t \in \Gamma'$, or $t.s \in \Gamma'$; so $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s = t$, or $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$. So Γ' is a scapegoat set for AMφ.

Similarly for AMβ.

C(I) We construct an interpretation $I = \langle W, R, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ', except that if $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s = t$, then $w_s = w_t$ (we might do this, in the usual way, by beginning with equivalence classes on subscripts). Then set $\langle w_s, w_t\rangle \in R$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$ and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} p_s$.

L2.8 If Γ_0 is consistent then for $\langle W, R, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L2.4, Γ' is s-maximal. By L2.6 and L2.7, Γ' is consistent and a scapegoat set. Now by induction on the number of operators in A_s,

Basis: If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

Assp: For any i, $0 \leq i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

If A_s has k operators, then it is of the form $\neg P_s$, $(P \supset Q)_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \equiv Q)_s$, $\Box P_s$, $\Diamond P_s$, $[\mathrm{F}]P_s$, $\langle \mathrm{F}\rangle P_s$, $[\mathrm{P}]P_s$, or $\langle \mathrm{P}\rangle P_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TK($\neg$), $v_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$; then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} P_s$; so by assumption, $v_{w_s}(P) = 0$; so by TK($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

($\supset$) A_s is $(P \supset Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} A_s$; then $v_{w_s}(P \supset Q) = 1$ but $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} (P \supset Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg(P \supset Q)_s$; from this it follows, by simple derivations, that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_s$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg Q_s$; so by consistency, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} Q_s$; so by assumption, $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by TK($\supset$), $v_{w_s}(P \supset Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} (P \supset Q)_s$ but $v_{w_s}(P \supset Q) = 0$. From the latter, by TK($\supset$), $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by assumption, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_s$ and $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} Q_s$; but since $\Gamma' \vdash^*_{NK^{(t)}_\alpha} (P \supset Q)_s$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_s$, by ($\supset$E), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} Q_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$, then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

($\wedge$)

($\vee$)

($\equiv$)

($\Box$) A_s is $\Box P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} A_s$; then $v_{w_s}(\Box P) = 1$ but $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} \Box P_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg\Box P_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg P_t$; from the first, by construction, $\langle w_s, w_t\rangle \in R$; and from the second, by consistency, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} P_t$; so by assumption, $v_{w_t}(P) = 0$; but $w_s R w_t$; so by TK($\Box$), $v_{w_s}(\Box P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \Box P_s$ but $v_{w_s}(\Box P) = 0$. From the latter, by TK($\Box$), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} P_t$; but since $w_s R w_t$, by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$; so by ($\Box$E), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NK^{(t)}_\alpha} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

($[\mathrm{F}]$)

($[\mathrm{P}]$) A_s is $[\mathrm{P}]P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} A_s$; then $v_{w_s}([\mathrm{P}]P) = 1$ but $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} [\mathrm{P}]P_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg[\mathrm{P}]P_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg P_t$; from the first, by construction, $\langle w_t, w_s\rangle \in R$; and from the second, by consistency, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} P_t$; so by assumption, $v_{w_t}(P) = 0$; but $w_t R w_s$; so by TK($[\mathrm{P}]$), $v_{w_s}([\mathrm{P}]P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} [\mathrm{P}]P_s$ but $v_{w_s}([\mathrm{P}]P) = 0$. From the latter, by TK($[\mathrm{P}]$), there is some $w_t \in W$ such that $w_t R w_s$ and $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NK^{(t)}_\alpha} P_t$; but since $w_t R w_s$, by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$;

so by ([P]E), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NK^{(t)}_\alpha} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

($\Diamond$) A_s is $\Diamond P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\Diamond P) = 1$; so by TK($\Diamond$), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 1$; so by assumption, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_t$; but since $w_s R w_t$, by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$; so by ($\Diamond$I), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \Diamond P_s$; so $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.
(ii)Suppose $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$; then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \Diamond P_s$; so by (MN), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg\Box\neg P_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg\neg P_t$; so by (DN), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_t$; so by assumption, $v_{w_t}(P) = 1$; but $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$; so by construction, $w_s R w_t$; so by TK($\Diamond$), $v_{w_s}(\Diamond P) = 1$; so $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

(⟨F⟩)

(⟨P⟩) A_s is $\langle\mathrm{P}\rangle P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\langle\mathrm{P}\rangle P) = 1$; so by TK(⟨P⟩), there is some $w_t \in W$ such that $w_t R w_s$ and $v_{w_t}(P) = 1$; so by assumption, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_t$; but since $w_t R w_s$, by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$; so by (⟨P⟩I), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \langle\mathrm{P}\rangle P_s$; so $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.
(ii)Suppose $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$; then $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \langle\mathrm{P}\rangle P_s$; so by (TMN), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg[\mathrm{P}]\neg P_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg\neg P_t$; so by (DN), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} P_t$; so by assumption, $v_{w_t}(P) = 1$; but $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$; so by construction, $w_t R w_s$; so by TK(⟨P⟩), $v_{w_s}(\Diamond P) = 1$; so $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_s$.

L2.9 If Γ_0 is consistent, then $\langle W, R, v\rangle$ constructed as above is a $K^{(t)}_\alpha$ interpretation.

In each case, we need to show that the interpretation meets the condition(s) α. Suppose Γ_0 is consistent.

(η) Suppose α includes condition η and $w_s \in W$. Then, by construction, s is a subscript in Γ'; so by reasoning as follows,

$$\begin{array}{r|lll}
1 & \Gamma' & & \\
2 & \quad\vert\ \underline{s.t} & & \text{A } (g,\ \text{AM}\eta) \\
3 & \quad\vert\ \top_t & & \top \text{ is a tautology} \\
4 & \quad\vert\ \Diamond\top_s & & \text{2,3 } \Diamond\text{I} \\
5 & \Diamond\top_s & & \text{2-4 AM}\eta \\
6 & \neg\Box\neg\top_s & & \text{5 MN}
\end{array}$$

$\Gamma' \vdash^*_{NK^{(t)}_\alpha} \neg\Box\neg\top_s$; but by L2.7, Γ' is a scapegoat set; so there is a t such that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$; so by construction, $\langle w_s, w_t\rangle \in R$ and η is satisfied.

(η')

(ρ) Suppose α includes condition ρ and $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.s$; so by construction, $\langle w_s, w_s\rangle \in R$ and ρ is satisfied.

(σ) Suppose α includes condition σ and $\langle w_s, w_t\rangle \in R$. Then by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$ so by (AMσ), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$; so by construction, $\langle w_t, w_s\rangle \in R$ and σ is satisfied.

(τ) Suppose α includes condition τ and $\langle w_s, w_t\rangle, \langle w_t, w_u\rangle \in R$. Then by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.u$; so by (AM$\tau$), $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.u$; so by construction, $\langle w_s, w_u\rangle \in R$ and τ is satisfied.

(δ) Suppose α includes condition δ and $\langle w_s, w_t\rangle \in R$; then by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$; so, since Γ' is a AMδ scapegoat set, there is a u such that $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.u$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} u.t$; so by construction, $\langle w_s, w_u\rangle \in R$ and $\langle w_u, w_t\rangle \in R$. So AM$\delta$ is satisfied.

(φ) Suppose α includes condition φ and $\langle w_r, w_s\rangle, \langle w_r, w_t\rangle \in R$; then by construction, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} r.s$ and $\Gamma' \vdash^*_{NK^{(t)}_\alpha} r.t$; so, since Γ' is a AMφ scapegoat set, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s.t$, $\Gamma' \vdash^*_{NK^{(t)}_\alpha} s = t$, or $\Gamma' \vdash^*_{NK^{(t)}_\alpha} t.s$; so by construction, $\langle w_s, w_t\rangle \in R$, $w_s = w_t$, or $\langle w_t, w_s\rangle \in R$. So AM$\varphi$ is satisfied.

(β)

Map For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L2.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Suppose Γ_0 is consistent and $A_0 \in \Gamma_0$; then by construction, $A_0 \in \Gamma'$; so $\Gamma' \vdash^*_{NK^{(t)}_\alpha} A_0$; so since Γ_0 is consistent, by L2.8, $v_{w_0}(A) = 1$. And similarly for any $A_0 \in \Gamma_0$. But $m(0) = w_0$; so $v_m(\Gamma_0) = 1$.

Main result: Suppose $\Gamma \models_{K^{(t)}_\alpha} A$ but $\Gamma \nvdash_{NK^{(t)}_\alpha} A$. Then $\Gamma_0 \models^*_{K^{(t)}_\alpha} A_0$ but $\Gamma_0 \nvdash^*_{NK^{(t)}_\alpha} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NK^{(t)}_\alpha} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NK^{(t)}_\alpha} A_0$; so $\Gamma_0 \nvdash^*_{NK^{(t)}_\alpha} \neg\neg A_0$; so by L2.2, $\Gamma_0 \cup \{\neg A_0\}$ is consistent; so by L2.9 and L2.10, there is a $K^{(t)}_\alpha$ interpretation $\langle W, R, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_0\}) = 1$; so $v_{m(0)}(\neg A) = 1$; so by TK($\neg$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VKα^*, $\Gamma_0 \not\models^*_{K^{(t)}_\alpha} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{K^{(t)}_\alpha} A$, then $\Gamma \vdash_{NK^{(t)}_\alpha} A$.

The argument for *NKv* is similar, and so omitted.

3 Non-Normal Modal Logics: *N*α, *L*α (ch. 4)

3.1 Language / Semantic Notions

LXα Allow X to be either N or L, depending on context, where for both $N\alpha$ and $L\alpha$, the basic language is the same as for $K\alpha$. The VOCABULARY consists of propositional parameters $p_0, p_1 \dots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\Box$ and $\Diamond$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$ and $\Diamond A$. In addition, we introduce $(A \,⥽\, B)$ as an abbreviation for $\Box(A \supset B)$.

IXα An INTERPRETATION is $\langle W, N, R, v\rangle$ where $N \subseteq W$. N is the set of *normal* worlds. Constraints on access are as for $K\alpha$. Thus, where α is empty or indicates some combination of the following constraints,

η	For any x, there is a y such that xRy	extendability
ρ	for all x, xRx	reflexivity
σ	for all x, y, if xRy then yRx	symmetry
τ	for all x, y, z, if xRy and yRz then xRz	transitivity

$\langle W, N, R, v\rangle$ is an $X\alpha$ interpretation when R meets the constraints from α.

INα Furthermore, an $N\alpha$ interpretation, specifically, is one in which v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$, as usual, but for any $w \notin N$ and P of the form $\Box A$, $v_w(P) = 0$; and for any $w \notin N$ and P of the form $\Diamond A$, $v_w(P) = 1$.

ILα However, an $L\alpha$ interpretation, specifically, is one in which v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$, as usual, but for any $w \notin N$ and P of the form $\Box A$ or $\Diamond A$, $v_w(P) = 1$ or $v_w(P) = 0$ (Truth values for modal formulas are arbitrarily assigned at non-normal worlds).

TX then applies for expressions not assigned a value directly.

TX For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\supset$) $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.

($\equiv$) $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.

($\Diamond$) $v_w(\Diamond A) = 1$ ($w \in N$) if some $x \in W$ such that wRx has $v_x(A) = 1$, and 0 otherwise.

($\Box$) $v_w(\Box A) = 1$ ($w \in N$) if all $x \in W$ such that wRx have $v_x(A) = 1$, and 0 otherwise.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VXα $\Gamma \models_{X\alpha} A$ iff there is no $X\alpha$ interpretation $\langle W, N, R, v\rangle$ and $w \in N$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

3.2 Natural Derivations: $\boldsymbol{NN}\alpha$, $\boldsymbol{NL}\alpha$

All the rules are as in $NK\alpha$ except that, for $NN\alpha$, whenever a subscript $s.t$ is introduced for $\Box$I or $\Diamond$E, either s is 0 or there is an additional premise of the sort $\Box A_s$ or $\neg\Diamond A_s$; and, for $NL\alpha$, whenever $s.t$ is introduced for $\Box$I, $\Box$E, $\Diamond$I, or $\Diamond$E, s is just 0. The resulting change on these rules is small.

$NN\alpha$

$$\Box\mathbf{I}_{N\alpha}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ \\ P_t \end{array} \\ \Box P_s \end{array}$$

where s is 0 or appears in some accessible $\Box A_s$ or $\neg\Diamond A_s$; and t does not appear in any undischarged premise or assumption

$$\Diamond\mathbf{E}_{N\alpha}\ \begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} s.t \\ P_t \\ \hline \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

where s is 0 or appears in some accessible $\Box A_s$ or $\neg\Diamond A_s$; and t does not appear in any undischarged premise or assumption and is not u

$NL\alpha$

$\Box$I$_{L\alpha}$

$\mid\ \mid\ s.t$

$\mid\ \mid\ P_t$

$\mid\ \Box P_s$

where s is 0; and t does not appear in any undischarged premise or assumption

$\Box$E$_{L\alpha}$

$\mid\ \Box P_s$

$\mid\ s.t$

$\mid\ P_t$

where s is 0

$\Diamond$I$_{L\alpha}$

$\mid\ P_t$

$\mid\ s.t$

$\mid\ \Diamond P_s$

where s is 0

$\Diamond$E$_{L\alpha}$

$\mid\ \Diamond P_s$

$\mid\ \mid\ s.t$

$\mid\ \mid\ P_t$

$\mid\ \mid\ Q_u$

$\mid\ Q_u$

where s is 0; and t does not appear in any undischarged premise or assumption and is not u

Derived rules carry over from $NK\alpha$. Note that MN remains as well (but restricted to subscript 0 in the L systems). In addition, the following are derived rules for ⥽I and ⥽E in $NK\alpha$, $NN\alpha$ and $NL\alpha$.

⥽I

$\mid\ \mid\ s.t$

$\mid\ \mid\ P_t$

$\mid\ \mid\ Q_t$

$\mid\ (P$ ⥽ $Q)_s$

constraints on s and t as for the corresponding NL, NN or NK $\Box$I rule.

⥽E

$\mid\ (P$ ⥽ $Q)_s$

$\mid\ s.t$

$\mid\ P_t$

$\mid\ Q_t$

Examples. We exhibit the new restrictions by considering derivations to show one part of MN, that $\Diamond P_s \vdash_{NN\alpha} \neg\Box\neg P_s$. In the case where $s \neq 0$, the derivation on the left violates the NN restriction on $\Diamond$E in its last line.

1	$\mid\ \Diamond P_s$	P
2	$\mid\ \mid\ s.t$	A (g, 1 $\Diamond$E)
3	$\mid\ \mid\ P_t$	
4	$\mid\ \mid\ \mid\ \Box\neg P_s$	A (c, $\neg$I)
5	$\mid\ \mid\ \mid\ \neg P_t$	2,4 $\Box$E
6	$\mid\ \mid\ \mid\ P_t$	3 R
7	$\mid\ \mid\ \neg\Box\neg P_s$	4-6 $\neg$I
8	$\mid\ \neg\Box\neg P_s$	1,2-7 $\Diamond$E

1	$\mid\ \Diamond P_s$	P
2	$\mid\ \mid\ \Box\neg P_s$	A (c, $\neg$I)
3	$\mid\ \mid\ \mid\ s.t$	A (g, 1 $\Diamond$E)
4	$\mid\ \mid\ \mid\ P_t$	
5	$\mid\ \mid\ \mid\ \mid\ \Box\neg P_s$	A (c, $\neg$I)
6	$\mid\ \mid\ \mid\ \mid\ \neg P_t$	3,5 $\Box$E
7	$\mid\ \mid\ \mid\ \mid\ P_t$	4 R
8	$\mid\ \mid\ \mid\ \neg\Box\neg P_s$	5-7 $\neg$I
9	$\mid\ \mid\ \neg\Box\neg P_s$	2,1,3-8 $\Diamond$E
10	$\mid\ \mid\ \Box\neg P_s$	2 R
11	$\mid\ \neg\Box\neg P_s$	2-10 $\neg$I

Supposing s is 0, each derivation is fine in *NN* and *NL*. However, if s is other than 0, on the left, (8) is automatically bad in *NL* and violates the *NN* restriction on $\Diamond$E, insofar as there is no accessible $\Box P_s$ or $\neg\Diamond P_s$. On the right, the derivation works in *NN* even when $s \neq 0$, insofar as we make the assumption for $\neg$I prior to that for $\Diamond$E. Note that, in this case, we *cite* the line with $\Box A_s$ for $\Diamond$E. Insofar as $s \neq 0$, the derivation would not do for *NL*.

3.3 Soundness and Completeness

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, N, R, v\rangle$, let m be a map from subscripts into W such that $m(0)$ is some member of N. Say $\langle W, N, R, v\rangle_m$ is $\langle W, N, R, v\rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VXα^* $\Gamma \models^*_{X\alpha} A_s$ $[\![s.t]\!]$ iff there is no $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$.

NXα^* $\Gamma \vdash^*_{NX\alpha} A_s$ $[\![s.t]\!]$ iff there is an $NX\alpha$ derivation of A_s $[\![s.t]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $s.t$). This is obvious for NXα^*. In the other case, there is a $\langle W, N, R, v\rangle_m$ and $w \in N$ that makes all the members of Γ_0 true and A_0 false just in case there *is* a world in N that makes the unsubscripted members of Γ true and A false. For the following, cases omitted are like ones worked, and so left to the reader.

Theorem 3.1 *NXα is sound: If* $\Gamma \vdash_{NX\alpha} A$ *then* $\Gamma \models_{X\alpha} A$.

L3.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{X\alpha} P_s$ $[\![s.t]\!]$, then $\Gamma' \models^*_{X\alpha} P_s$ $[\![s.t]\!]$.

Reasoning parallel to that for L2.1 of *NKα*.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NX\alpha} \mathcal{P}$ then $\Gamma \models^*_{X\alpha} \mathcal{P}$. Suppose $\Gamma \vdash^*_{NX\alpha} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{X\alpha} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![s.t]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{s.t\}]\!]$; so for any $\langle W, N, R, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $[\![\langle m(s), m(t)\rangle \in R]\!]$; so there is no $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$. So by VX$\alpha^*$, $\Gamma_1 \models^*_{X\alpha} A_s$ $[\![s.t]\!]$, where this is just to say, $\Gamma_1 \models^*_{X\alpha} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^*_{X\alpha} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I, $\equiv$E, $\Box$I$_{N_\alpha}$, $\Box$E, $\Diamond$I, $\Diamond$E$_{N_\alpha}$, $\Box$I$_{L_\alpha}$, $\Box$E$_{L_\alpha}$, $\Diamond$I$_{L_\alpha}$, $\Diamond$E$_{L_\alpha}$ or, depending on the system, AMη, AMρ, AMσ or AMτ. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\supset$I)

($\supset$E) If $\mathcal{P}_k$ arises by $\supset$E, then the picture is like this,

$$\begin{array}{r|l} i & (A \supset B)_s \\ j & A_s \\ & \\ k & B_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is B_s. By assumption, $\Gamma_i \models^*_{X\alpha} (A \supset B)_s$ and $\Gamma_j \models^*_{X\alpha} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{X\alpha} (A \supset B)_s$ and $\Gamma_k \models^*_{X\alpha} A_s$. Suppose $\Gamma_k \not\models^*_{X\alpha} B_s$; then by VX$\alpha^*$, there is some $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VXα^*, $v_{m(s)}(A \supset B) = 1$ and $v_{m(s)}(A) = 1$; from the former, by TX($\supset$), $v_{m(s)}(A) = 0$ or $v_{m(s)}(B) = 1$; so $v_{m(s)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} B_s$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

($\wedge$I)

($\wedge$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & \underline{A_s} \\
 & & \\
i & & B_t \\
j & & \neg B_t \\
k & \multicolumn{2}{l}{\neg A_s}
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{X\alpha} B_t$ and $\Gamma_j \models^*_{X\alpha} \neg B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L3.1, $\Gamma_k \cup \{A_s\} \models^*_{X\alpha} B_t$ and $\Gamma_k \cup \{A_s\} \models^*_{X\alpha} \neg B_t$. Suppose $\Gamma_k \not\models^*_{X\alpha} \neg A_s$; then by VX$\alpha^*$, there is an $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TX($\neg$), $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by VX$\alpha^*$, $v_{m(t)}(B) = 1$ and $v_{m(t)}(\neg B) = 1$; from the latter, by TX($\neg$), $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} \neg A_s$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

($\neg$E)

($\vee$I)

($\vee$E)

($\equiv$I)

($\equiv$E)

($\Box$I$_{N\alpha}$) If $\mathcal{P}_k$ arises by $\Box$I$_{N\alpha}$, then the picture is like this,

$$\begin{array}{r|l|l}
 & & \underline{s.t} \\
 & & \\
j & & A_t \\
k & \multicolumn{2}{l}{\Box A_s}
\end{array}$$

where $j < k$, s is 0 or introduced in some accessible $\Box P_s$ or $\neg\Diamond P_s$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\Box A_s$. By assumption, $\Gamma_j \models^*_{X\alpha} A_t$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{s.t\}$; so by L3.1, $\Gamma_k \cup \{s.t\} \models^*_{X\alpha} A_t$. Suppose $\Gamma_k \not\models^*_{X\alpha} \Box A_s$; then by VX$\alpha^*$, there is an $X\alpha$ interpretation, specifically an $N\alpha$ interpretation, $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\Box A) = 0$. If s is 0, then $m(s) \in N$; if s is introduced in some $\Box P_s$ on accessible line j, then by assumption, $\Gamma_j \models^*_{X\alpha} \Box P_s$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{X\alpha} \Box P_s$; so by VX$\alpha^*$, $v_{m(s)}(\Box P) = 1$; so, since $\langle W, N, R, v\rangle_m$ is an $N\alpha$ interpretation, $m(s) \in N$; if s is introduced in some $\neg\Diamond P_s$ on an accessible line

j, then by assumption, $\Gamma_j \models^*_{X\alpha} \neg\Diamond P_s$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{X\alpha} \neg\Diamond P_s$; so by VX$\alpha^*$, $v_{m(s)}(\neg\Diamond P) = 1$; so by TX($\neg$), $v_{m(s)}(\Diamond P) = 0$; so, since $\langle W, N, R, v\rangle_m$ is an $N\alpha$ interpretation, $m(s) \in N$; in any case, then, $m(s) \in N$. So by TX($\Box$), there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VX$\alpha^*$, $v_{m'(t)}(A) = 1$. But $m'(t) = w$; so $v_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} \Box A_s$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

($\Box$E) If $\mathcal{P}_k$ arises by $\Box$E, then the picture is like this,

$$\begin{array}{r|l} i & \Box A_s \\ j & s.t \\ & \\ k & A_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{X\alpha} \Box A_s$ and $\Gamma_j \models^*_{X\alpha} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{X\alpha} \Box A_s$ and $\Gamma_k \models^*_{X\alpha} s.t$. Suppose $\Gamma_k \not\models^*_{X\alpha} A_t$; then by VX$\alpha^*$, there is some $X\alpha$ interpretation, specifically an $N\alpha$ interpretation, $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VXα^*, $v_{m(s)}(\Box A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, since $\langle W, N, R, v\rangle_m$ is an $N\alpha$ interpretation, $m(s) \in N$, and so, by TX($\Box$), any w such that $m(s)Rw$ has $v_w(A) = 1$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} A_t$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

($\Diamond$I)

($\Diamond$E$_{N\alpha}$) If $\mathcal{P}_k$ arises by $\Diamond$E$_{N\alpha}$, then the picture is like this,

$$\begin{array}{r|c|l} i & \multicolumn{2}{l}{\Diamond A_s} \\ & & A_t \\ & & s.t \\ \cline{3-3} & & \\ j & & B_u \\ k & \multicolumn{2}{l}{B_u} \end{array}$$

where $i, j < k$, s is 0 or introduced in some accessible $\Box P_s$ or $\neg\Diamond P_s$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \models^*_{X\alpha} \Diamond A_s$ and $\Gamma_j \models^*_{X\alpha} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t, s.t\}$; so by L3.1, $\Gamma_k \models^*_{X\alpha} \Diamond A_s$ and $\Gamma_k \cup \{A_t, s.t\} \models^*_{X\alpha} B_u$. Suppose $\Gamma_k \not\models^*_{X\alpha} B_u$; then by VX$\alpha^*$, there is an $X\alpha$ interpretation, specifically an $N\alpha$ interpretation, $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$. If s is 0, then $m(s) \in N$; if s is introduced in some $\Box P_s$ on accessible line h, then by assumption, $\Gamma_h \models^*_{X\alpha} \Box P_s$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{X\alpha} \Box P_s$; so by VX$\alpha^*$, $v_{m(s)}(\Box P) = 1$; so, since $\langle W, N, R, v\rangle_m$ is an $N\alpha$ interpretation, $m(s) \in N$; if s is introduced in some $\neg\Diamond P_s$ on an accessible line h, then by assumption, $\Gamma_h \models^*_{X\alpha} \neg\Diamond P_s$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{X\alpha} \neg\Diamond P_s$; so by VX$\alpha^*$, $v_{m(s)}(\neg\Diamond P) = 1$; so by TX($\neg$), $v_{m(s)}(\Diamond P) = 0$; so, since $\langle W, N, R, v\rangle_m$ is an $N\alpha$ interpretation, $m(s) \in N$; in any case, then, $m(s) \in N$. Since $v_m(\Gamma_k) = 1$, by VXα^*, $v_{m(s)}(\Diamond A) = 1$; so, since $m(s) \in N$, by TX($\Diamond$), there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $v_{m'(t)}(A) = 1$ and $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{A_t, s.t\}) = 1$; so by VX$\alpha^*$, $v_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} B_u$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

($\Box$I$_{L\alpha}$) If $\mathcal{P}_k$ arises by $\Box$I$_{L\alpha}$, then the picture is like this,

$$\begin{array}{r|l|l}
 & & s.t \\ \cline{3-3}
 & & \\
j & & A_t \\
k & \multicolumn{2}{l}{\Box A_s}
\end{array}$$

where $j < k$, s is 0, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\Box A_s$. By assumption, $\Gamma_j \models^*_{X\alpha} A_t$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{s.t\}$; so by L3.1, $\Gamma_k \cup \{s.t\} \models^*_{X\alpha} A_t$. Suppose $\Gamma_k \not\models^*_{X\alpha} \Box A_s$; then by VX$\alpha^*$, there is an $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\Box A) = 0$. Since s is 0, $m(s) \in N$. So by TX($\Box$), there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VX$\alpha^*$, $v_{m'(t)}(A) = 1$. But $m'(t) = w$; so $v_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{X\alpha} \Box A_s$, which is to say, $\Gamma_k \vDash^*_{X\alpha} \mathcal{P}_k$.

($\Box$E$_{L\alpha}$) If $\mathcal{P}_k$ arises by $\Box$E$_{L\alpha}$, then the picture is like this,

$$\begin{array}{r|l} i & \Box A_s \\ j & s.t \\ & \\ k & A_t \end{array}$$

where $i, j < k$, s is 0, and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \vDash^*_{X\alpha} \Box A_s$ and $\Gamma_j \vDash^*_{X\alpha} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \vDash^*_{X\alpha} \Box A_s$ and $\Gamma_k \vDash^*_{X\alpha} s.t$. Suppose $\Gamma_k \nvDash^*_{X\alpha} A_t$; then by VX$\alpha^*$, there is some $X\alpha$ interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VXα^*, $v_{m(s)}(\Box A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, since s is 0 and so $m(s) \in N$, by TX($\Box$), any w such that $m(s)Rw$ has $v_w(A) = 1$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{X\alpha} A_t$, which is to say, $\Gamma_k \vDash^*_{X\alpha} \mathcal{P}_k$.

($\Diamond$I$_{L\alpha}$)

($\Diamond$E$_{L\alpha}$) If $\mathcal{P}_k$ arises by $\Diamond$E$_{L\alpha}$, then the picture is like this,

$$\begin{array}{r|c|l} i & \multicolumn{2}{l}{\Diamond A_s} \\ & & A_t \\ & & s.t \\ \cline{3-3} & & \\ & & \\ j & & B_u \\ k & \multicolumn{2}{l}{B_u} \end{array}$$

where $i, j < k$, s is 0, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \vDash^*_{X\alpha} \Diamond A_s$ and $\Gamma_j \vDash^*_{X\alpha} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t, s.t\}$; so by L3.1, $\Gamma_k \vDash^*_{X\alpha} \Diamond A_s$ and $\Gamma_k \cup \{A_t, s.t\} \vDash^*_{X\alpha} B_u$. Suppose $\Gamma_k \nvDash^*_{X\alpha} B_u$; then by VX$\alpha^*$, there is an $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$. Since s is 0, $m(s) \in N$. Since $v_m(\Gamma_k) = 1$, by VXα^*, $v_{m(s)}(\Diamond A) = 1$; so, since $m(s) \in N$, by TX($\Diamond$), there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $v_{m'(t)}(A) = 1$ and $\langle m'(s), m'(t)\rangle \in R$;

so $v_{m'}(\Gamma_k \cup \{A_t, s.t\}) = 1$; so by $\mathrm{VX}\alpha^*$, $v_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} B_u$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

(AMη) If $\mathcal{P}_k$ arises by AMη, then the picture is like this,

$$\begin{array}{r|l|l} & & s.t \\ \cline{3-3} & & \\ j & & A_u \\ k & \multicolumn{2}{l}{A_u} \end{array}$$

where $j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is A_u. Where this rule is included in $NX\alpha$, $X\alpha$ includes condition η. By assumption, $\Gamma_j \models^*_{X\alpha} A_u$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{s.t\}$; so by L3.1, $\Gamma_k \cup \{s.t\} \models^*_{X\alpha} A_u$. Suppose $\Gamma_k \not\models^*_{X\alpha} A_u$; then by $\mathrm{VX}\alpha^*$, there is an $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(A) = 0$. By condition η, there is a $w \in W$ such that $m(s)Rw$; consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by $\mathrm{VX}\alpha^*$, $v_{m'(u)}(A) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} A_u$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

(AMρ)

(AMσ)

(AMτ) If $\mathcal{P}_k$ arises by AMτ, then the picture is like this,

$$\begin{array}{r|l} i & s.t \\ j & t.u \\ & \\ k & s.u \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $s.u$. Where this rule is in $NX\alpha$, $X\alpha$ includes condition τ. By assumption, $\Gamma_i \models^*_{X\alpha} s.t$ and $\Gamma_j \models^*_{X\alpha} t.u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L3.1, $\Gamma_k \models^*_{X\alpha} s.t$ and $\Gamma_k \models^*_{X\alpha} t.u$. Suppose $\Gamma_k \not\models^*_{X\alpha} s.u$; then by $\mathrm{VX}\alpha^*$, there is some $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(u)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by $\mathrm{VX}\alpha^*$, $\langle m(s), m(t)\rangle \in R$ and $\langle m(t), m(u)\rangle \in R$; and by condition τ, for any $\langle x, y\rangle, \langle y, z\rangle \in$

R, $\langle x, z\rangle \in R$; so $\langle m(s), m(u)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{X\alpha} s.u$, which is to say, $\Gamma_k \models^*_{X\alpha} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{X\alpha} \mathcal{P}_i$.

THEOREM 3.2 *$NX\alpha$ is complete: if $\Gamma \models_{X\alpha} A$ then $\Gamma \vdash_{NX\alpha} A$.*

Suppose $\Gamma \models_{X\alpha} A$; then $\Gamma_0 \models^*_{X\alpha} A_0$; we show that $\Gamma_0 \vdash^*_{NX\alpha} A_0$. Again, this reduces to the standard notion. For the following, fix on some particular constraint(s) α. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NX\alpha} A_s$ and $\Gamma \vdash^*_{NX\alpha} \neg A_s$.

L3.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NX\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^*_{NX\alpha} \neg P_s$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^*_{NX\alpha} A_t$ and $\Gamma \cup \{P_s\} \vdash^*_{NX\alpha} \neg A_t$. But then we can argue,

1	Γ	
2	$\quad P_s$	A (c, $\neg$I)
3	$\quad A_t$	from $\Gamma \cup \{P_s\}$
4	$\quad \neg A_t$	from $\Gamma \cup \{P_s\}$
5	$\neg P_s$	2-4 $\neg$I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^*_{NX\alpha} \neg P_s$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NX\alpha} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

L3.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1$ $\mathcal{P}_2 \ldots$

Proof by construction as for L2.3 of *$NK\alpha$*.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NX\alpha} A_s$ or $\Gamma \vdash^*_{NX\alpha} \neg A_s$.

SGT$_{N\alpha}$ Γ is an $N\alpha$ SCAPEGOAT set iff for every formula of the form $(\Box P \wedge \neg\Box A)_s$, if $\Gamma \vdash^*_{NN\alpha} (\Box P \wedge \neg\Box A)_s$ then there is some t such that $\Gamma \vdash^*_{NN\alpha} s.t$ and $\Gamma \vdash^*_{NN\alpha} \neg A_t$.

SGT$_{L\alpha}$ Γ is an $L\alpha$ SCAPEGOAT set iff for every formula of the form $\neg\Box A_0$, if $\Gamma \vdash^*_{NL\alpha} \neg\Box A_0$ then there is some t such that $\Gamma \vdash^*_{NL\alpha} 0.t$ and $\Gamma \vdash^*_{NL\alpha} \neg A_t$.

$C(\Gamma')_{N\alpha}$ For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L3.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

(i) $\Omega_i = \Omega_{i-1}$ if $\Omega_{i-1} \vdash^*_{NN\alpha} \neg A_s$
$\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$ if $\Omega_{i-1} \nvdash^*_{NN\alpha} \neg A_s$
and
(ii) $\Omega_i = \Omega_{i^*}$ if A_s is not of the form $(\Box Q \wedge \neg\Box P_s)$
(iii) $\Omega_i = \Omega_{i^*} \cup \{s.t, \neg P_t\}$ if A_s is of the form $(\Box Q \wedge \neg\Box P)_s$
-where t is the first subscript not included in Ω_{i^*}
then
$\Gamma' = \bigcup_{i \geq 0} \Omega_i$

$C(\Gamma')_{L\alpha}$ For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L3.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

(i) $\Omega_i = \Omega_{i-1}$ if $\Omega_{i-1} \vdash^*_{NL\alpha} \neg A_s$
$\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$ if $\Omega_{i-1} \nvdash^*_{NL\alpha} \neg A_s$
and
(ii) $\Omega_i = \Omega_{i^*}$ if A_s is not of the form $\neg\Box P_0$
(iii) $\Omega_i = \Omega_{i^*} \cup \{0.t, \neg P_t\}$ if A_s is of the form $\neg\Box P_0$
-where t is the first subscript not included in Ω_{i^*}
then
$\Gamma' = \bigcup_{i \geq 0} \Omega_i$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L3.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NX\alpha} A_s$ and $\Gamma' \nvdash^*_{NX\alpha} \neg A_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NX\alpha} \neg A_s$ then $\Gamma' \vdash^*_{NX\alpha} \neg A_s$; but $\Gamma' \nvdash^*_{NX\alpha} \neg A_s$; so $\Omega_{i-1} \nvdash^*_{NX\alpha} \neg A_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NX\alpha} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L3.5$_{N\alpha}$ If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, or (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$ or (iii) $\Omega_{k^*} \cup \{s.t, \neg P_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NN\alpha} \neg A_s$; so by L3.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, \neg P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $(\Box Q \wedge \neg \Box P)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{s.t, \neg P_t\} \vdash^*_{NN\alpha} A_u$ and $\Omega_{k^*} \cup \{s.t, \neg P_t\} \vdash^*_{NN\alpha} \neg A_u$. So reason as follows,

$$
\begin{array}{r|lll}
1 & \Omega_{k^*} & & \\
2 & (\Box Q \wedge \neg \Box P)_s & & \text{from } \Omega_{k^*} \\
3 & \Box Q_s & & 2\ \wedge\text{E} \\
4 & \mid\ s.t & & \text{A } (g, \Box\text{I}_{N\alpha}) \\
5 & \mid\ \mid\ \neg P_t & & \text{A } (c, \neg\text{E}) \\
6 & \mid\ \mid\ A_u & & \text{from } \Omega_{k^*} \cup \{s.t, \neg P_t\} \\
7 & \mid\ \mid\ \neg A_u & & \text{from } \Omega_{k^*} \cup \{s.t, \neg P_t\} \\
8 & \mid\ P_t & & 5\text{-}7\ \neg\text{E} \\
9 & \Box P_s & & 3\text{,}4\text{-}8\ \Box\text{I}_{N\alpha}
\end{array}
$$

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NN\alpha} \Box P_s$; but $(\Box Q \wedge \neg \Box P)_s \in \Omega_{k^*}$; so with ($\wedge$E), $\Omega_{k^*} \vdash^*_{NN\alpha} \neg \Box P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L3.5$_{L\alpha}$ If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, or (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$ or (iii) $\Omega_{k^*} \cup \{0.t, \neg P_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NL\alpha} \neg A_s$; so by L3.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{0.t, \neg P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg\Box P_0 \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{0.t, \neg P_t\} \vdash^*_{NL\alpha} A_u$ and $\Omega_{k^*} \cup \{0.t, \neg P_t\} \vdash^*_{NL\alpha} \neg A_u$. So reason as follows,

$$\begin{array}{r|lll}
1 & \Omega_{k^*} & & \\
2 & \quad \underline{|\; 0.t} & & \text{A } (g, \Box\text{I}_{L\alpha}) \\
3 & \quad |\quad \underline{|\; \neg P_t} & & \text{A } (c, \neg\text{E}) \\
4 & \quad |\quad |\; A_u & & \text{from } \Omega_{k^*} \cup \{0.t, \neg P_t\} \\
5 & \quad |\quad |\; \neg A_u & & \text{from } \Omega_{k^*} \cup \{0.t, \neg P_t\} \\
6 & \quad |\; P_t & & \text{3-5 } \neg\text{E} \\
7 & \Box P_s & & \text{2-6 } \Box\text{I}_{L\alpha}
\end{array}$$

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NL\alpha} \Box P_s$; but $\neg\Box P_0 \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NL\alpha} \neg\Box P_0$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L3.6 If Γ_0 is consistent, then Γ' is consistent.

Suppose Γ_0 is consistent, but Γ' is not; from the latter, there is some P_s such that $\Gamma' \vdash^*_{NX\alpha} P_s$ and $\Gamma' \vdash^*_{NX\alpha} \neg P_s$. Consider derivations D1 and D2 of these results, and the premises $\mathcal{P}_i \ldots \mathcal{P}_j$ of these derivations. By construction, there is an Ω_k with each of these premises as a member; so D1 and D2 are derivations from Ω_k; so Ω_k is not consistent. But since Γ_0 is consistent, by L3.5, Ω_k is consistent. This is impossible; reject the assumption: if Γ_0 is consistent then Γ' is consistent.

L3.7$_{N\alpha}$ If Γ_0 is consistent, then Γ' is an $N\alpha$ scapegoat set.

Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge \neg\Box P)_s$. By L3.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NN\alpha} \neg(\Box Q \wedge \neg\Box P)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{(\Box Q \wedge \neg\Box P)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, \neg P_t\}$; so by construction, $s.t \in \Gamma'$ and $\neg P_t \in \Gamma'$; so $\Gamma' \vdash^*_{NN\alpha} s.t$ and $\Gamma' \vdash^*_{NN\alpha} \neg P_t$. So Γ' is an $N\alpha$ scapegoat set.

L3.7$_{L\alpha}$ If Γ_0 is consistent, then Γ' is an $L\alpha$ scapegoat set.

Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NL\alpha} \neg\Box P_0$. By L3.6, Γ' is consistent; and subscript 0 is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NL\alpha} \neg\neg\Box P_0$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg\Box P_0\}$ and $\Omega_i = \Omega_{i^*} \cup \{0.t, \neg P_t\}$; so by construction, $0.t \in \Gamma'$ and $\neg P_t \in \Gamma'$; so $\Gamma' \vdash^*_{NL\alpha} 0.t$ and $\Gamma' \vdash^*_{NL\alpha} \neg P_t$. So Γ' is an $L\alpha$ scapegoat set.

C$(I)_{N\alpha}$ We construct an interpretation $I = \langle W, N, R, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $w_s \in N$ iff there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; for any $w_s \notin N$ and any A of the form $\Box P$ or $\Diamond P$, set $v_{w_s}(\Box P) = 0$ and $v_{w_s}(\Diamond P) = 1$; set $R = \{\langle w_s, w_s\rangle \mid w_s \in (W - N)\} \cup \{\langle w_s, w_t\rangle \mid \Gamma' \vdash^*_{NN\alpha} s.t\}$; and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} p_s$.

Note that $w_0 \in N$: By a simple derivation, $\vdash^*_{NX\alpha} \Box\top_0$; so $\Gamma' \vdash^*_{NX\alpha} \Box\top_0$; so $w_0 \in N$.

C$(I)_{L\alpha}$ We construct an interpretation $I = \langle W, N, R, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $w_s \in N$ iff s is 0; for any $w_s \notin N$ and any A of the form $\Box P$ or $\Diamond P$, set $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NL\alpha} A_s$; set $R = \{\langle w_s, w_s\rangle | w_s \in (W - N)\} \cup \{\langle w_s, w_t\rangle | \Gamma' \vdash^*_{NL\alpha} s.t\}$; and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NL\alpha} p_s$.

L3.8 If Γ_0 is consistent then for $\langle W, N, R, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L3.4, Γ' is s-maximal. By L3.6 and L3.7, Γ' is consistent and an $X\alpha$ scapegoat set. Now by induction on the number of operators in A_s, *Basis:* If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} A_s$.

Assp: For any i, $0 \leq i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} A_s$.

If A_s has k operators, then it is of the form $\neg P_s$, $(P \supset Q)_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \equiv Q)_s$, $\Box P_s$ or $\Diamond P_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TX($\neg$), $v_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NX\alpha} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NX\alpha} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NX\alpha} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NX\alpha} A_s$; then $\Gamma' \vdash^*_{NX\alpha} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NX\alpha} P_s$; so by assumption, $v_{w_s}(P) = 0$; so by TX($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} A_s$.

($\supset$) A_s is $(P \supset Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NX\alpha} A_s$; then $v_{w_s}(P \supset Q) = 1$ but $\Gamma' \nvdash^*_{NX\alpha} (P \supset Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NX\alpha} \neg(P \supset Q)_s$; from this it follows, by simple derivations, that $\Gamma' \vdash^*_{NX\alpha} P_s$ and $\Gamma' \vdash^*_{NX\alpha} \neg Q_s$; so by consistency, $\Gamma' \nvdash^*_{NX\alpha} Q_s$; so by assumption, $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by TX($\supset$), $v_{w_s}(P \supset Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NX\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NX\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NX\alpha} (P \supset Q)_s$ but $v_{w_s}(P \supset Q) = 0$. From the latter, by TX($\supset$), $v_{w_s}(P) = 1$ and $v_{w_s}(Q) = 0$; so by assumption, $\Gamma' \vdash^*_{NX\alpha} P_s$ and $\Gamma' \nvdash^*_{NX\alpha} Q_s$; but since $\Gamma' \vdash^*_{NX\alpha} (P \supset Q)_s$ and $\Gamma' \vdash^*_{NX\alpha} P_s$, by ($\supset$E), $\Gamma' \vdash^*_{NX\alpha} Q_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NX\alpha} A_s$, then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} A_s$.

($\wedge$)

($\vee$)

($\equiv$)

$(\Box)_{N\alpha}$ A_s is $\Box P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} A_s$; then $v_{w_s}(\Box P) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} \Box P_s$. From the former, by construction, $w_s \in N$; so by construction, there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; from the latter, by s-maximality, $\Gamma' \vdash^*_{NN\alpha} \neg\Box P_s$; so by ($\wedge$I), $\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge \neg\Box P)_s$; so, since Γ' is an $N\alpha$ scapegoat set, there is some t such that $\Gamma' \vdash^*_{NN\alpha} s.t$ and $\Gamma' \vdash^*_{NN\alpha} \neg P_t$; from the first, by construction, $\langle w_s, w_t \rangle \in R$; and from the second, by consistency, $\Gamma' \nvdash^*_{NN\alpha} P_t$; so by assumption, $v_{w_t}(P) = 0$; but $w_s R w_t$; so by TX($\Box$), $v_{w_s}(\Box P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NN\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NN\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NN\alpha} \Box P_s$ but $v_{w_s}(\Box P) = 0$. From the former, by construction, $w_s \in N$; so with the latter, by TX($\Box$), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NN\alpha} P_t$; but since $w_s R w_t$ and $w_s \in N$, by construction, $\Gamma' \vdash^*_{NN\alpha} s.t$; so by ($\Box$E$_{N\alpha}$), $\Gamma' \vdash^*_{NN\alpha} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NN\alpha} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

$(\Box)_{L\alpha}$ A_s is $\Box P_s$. If $w_s \notin N$, then by construction, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NL\alpha} A_s$. So suppose $w_s \in N$; then by construction, s is 0.
(i) Suppose $v_{w_0}(A) = 1$ but $\Gamma' \nvdash^*_{NL\alpha} A_0$; then $v_{w_0}(\Box P) = 1$ but $\Gamma' \nvdash^*_{NL\alpha} \Box P_0$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NL\alpha} \neg\Box P_0$; so, since Γ' is an $L\alpha$ scapegoat set, there is some t such that $\Gamma' \vdash^*_{NL\alpha} 0.t$ and $\Gamma' \vdash^*_{NL\alpha} \neg P_t$; from the first, by construction, $\langle w_0, w_t \rangle \in R$; and from the second, by consistency, $\Gamma' \nvdash^*_{NL\alpha} P_t$; so by assumption, $v_{w_t}(P) = 0$; but $w_0 R w_t$; so by TX($\Box$), $v_{w_0}(\Box P) = 0$. This is impossible; reject the assumption: if $v_{w_0}(A) = 1$, then $\Gamma' \vdash^*_{NL\alpha} A_0$.

(ii) Suppose $\Gamma' \vdash^*_{NL\alpha} A_0$ but $v_{w_0}(A) = 0$; then $\Gamma' \vdash^*_{NL\alpha} \Box P_0$ but $v_{w_0}(\Box P) = 0$. $w_0 \in N$; so with the latter, by TX($\Box$), there is some $w_t \in W$ such that $w_0 R w_t$ and $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NL\alpha} P_t$; but since $w_0 R w_t$ and $w_0 \in N$, by construction, $\Gamma' \vdash^*_{NL\alpha} 0.t$; so by ($\Box$E$_{L\alpha}$), $\Gamma' \vdash^*_{NL\alpha} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NL\alpha} A_0$ then $v_{w_0}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NL\alpha} A_s$.

$(\Diamond)_{N\alpha}$ A_s is $\Diamond P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} A_s$; then $v_{w_s}(\Diamond P) = 1$ but $\Gamma' \nvdash^*_{NN\alpha} \Diamond P_s$; from the latter, by s-maximality, $\Gamma' \vdash^*_{NN\alpha} \neg\Diamond P_s$; so by (MN), $\Gamma' \vdash^*_{NN\alpha} \Box\neg P_s$; so by construction, $w_s \in N$; so, with the former, by TX($\Diamond$), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 1$; so by assumption, $\Gamma' \vdash^*_{NN\alpha} P_t$; but since $w_s R w_t$ and $w_s \in N$, by construction, $\Gamma' \vdash^*_{NN\alpha} s.t$; so by ($\Diamond$I$_{N\alpha}$), $\Gamma' \vdash^*_{NN\alpha} \Diamond P_s$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NN\alpha} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NN\alpha} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NN\alpha} \Diamond P_s$ but $v_{w_s}(\Diamond P) = 0$. From the latter, by construction, $w_s \in N$;

so by construction, there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; from the former, by (MN), $\Gamma' \vdash^*_{NN\alpha} \neg\Box\neg P_s$; so by ($\wedge$I), $\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge \neg\Box\neg P)_s$; so, since Γ' is an $N\alpha$ scapegoat set, there is some t such that $\Gamma' \vdash^*_{NN\alpha} s.t$ and $\Gamma' \vdash^*_{NN\alpha} \neg\neg P_t$; from the first, by construction, $\langle w_s, w_t \rangle \in R$; from the second, by (DN), $\Gamma' \vdash^*_{NN\alpha} P_t$; so by assumption, $v_{w_t}(P) = 1$; so since $w_s R w_t$ by TX($\Diamond$), $v_{w_s}(\Diamond P) = 1$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NN\alpha} A_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NN\alpha} A_s$.

$(\Diamond)_{L\alpha}$ A_s is $\Diamond P_s$. If $w_s \notin N$, then by construction, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NL\alpha} A_s$. So suppose $w_s \in N$; then by construction, s is 0.

(i) Suppose $v_{w_0}(A) = 1$ but $\Gamma' \nvdash^*_{NL\alpha} A_0$; then $v_{w_0}(\Diamond P) = 1$ but $\Gamma' \nvdash^*_{NL\alpha} \Diamond P_0$; from the latter, by s-maximality, $\Gamma' \vdash^*_{NL\alpha} \neg\Diamond P_0$; Since $w_0 \in N$, with the former, by TX($\Diamond$), there is some $w_t \in W$ such that $w_0 R w_t$ and $v_{w_t}(P) = 1$; so by assumption, $\Gamma' \vdash^*_{NL\alpha} P_t$; but since $w_0 R w_t$ and $w_0 \in N$, by construction, $\Gamma' \vdash^*_{NL\alpha} 0.t$; so by ($\Diamond$I$_{L\alpha}$), $\Gamma' \vdash^*_{NL\alpha} \Diamond P_0$. This is impossible; reject the assumption: if $v_{w_0}(A) = 1$ then $\Gamma' \vdash^*_{NL\alpha} A_0$.

(ii) Suppose $\Gamma' \vdash^*_{NL\alpha} A_0$ but $v_{w_0}(A) = 0$; then $\Gamma' \vdash^*_{NL\alpha} \Diamond P_0$ but $v_{w_0}(\Diamond P) = 0$. From the former, by (MN), $\Gamma' \vdash^*_{NL\alpha} \neg\Box\neg P_0$; so, since Γ' is an $L\alpha$ scapegoat set, there is some t such that $\Gamma' \vdash^*_{NL\alpha} 0.t$ and $\Gamma' \vdash^*_{NL\alpha} \neg\neg P_t$; from the first, by construction, $\langle w_0, w_t \rangle \in R$; from the second, by (DN), $\Gamma' \vdash^*_{NL\alpha} P_t$; so by assumption, $v_{w_t}(P) = 1$; so since $w_0 R w_t$ by TX($\Diamond$), $v_{w_0}(\Diamond P) = 1$. This is impossible; reject the assumption: if $v_{w_0}(A) = 1$ then $\Gamma' \vdash^*_{NL\alpha} A_0$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NL\alpha} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NX\alpha} A_s$.

L3.9 If Γ_0 is consistent, then $\langle W, N, R, v \rangle$ constructed as above is an $X\alpha$ interpretation.

In each case, we need to show that the interpretation meets the condition(s) α. Suppose Γ_0 is consistent.

$(\eta)_{N\alpha}$ Suppose α includes condition η and $w_s \in W$. If $w_s \notin N$, then by construction, $\langle w_s, w_s \rangle \in R$ and η is satisfied. So suppose $w_s \in N$. Then by construction, there is some Q such that $\Gamma' \vdash^*_{NN\alpha} \Box Q_s$; so by reasoning as follows,

$$\begin{array}{r|l l}
1 & \Gamma' & \\
2 & \Box Q_s & \text{from } \Gamma' \\
3 & \quad \vert\ \underline{s.t} & \text{A } (g, \text{AM}\eta) \\
4 & \quad \vert\ \top_t & \top \text{ is a tautology} \\
5 & \quad \vert\ \Diamond \top_s & 3{,}4\ \Diamond\text{I} \\
6 & \Diamond \top_s & 3\text{-}5\ \text{AM}\eta \\
7 & \neg\Box\neg\top_s & 6\ \text{MN} \\
8 & (\Box Q \wedge \neg\Box\neg\top)_s & 2{,}7\ \wedge\text{I}
\end{array}$$

$\Gamma' \vdash^*_{NN\alpha} (\Box Q \wedge \neg\Box\neg\top)_s$; but by L3.7, Γ' is an $N\alpha$ scapegoat set; so there is a t such that $\Gamma' \vdash^*_{NN\alpha} s.t$; so by construction, $\langle w_s, w_t\rangle \in R$ and η is satisfied.

$(\eta)_{L\alpha}$ Suppose α includes condition η and $w_s \in W$. If $w_s \notin N$, then by construction, $\langle w_s, w_s\rangle \in R$ and η is satisfied. So suppose $w_s \in N$. Then by construction, s is 0; so by reasoning as follows,

$$\begin{array}{r|l l}
1 & \Gamma' & \\
2 & \quad \vert\ \underline{0.t} & \text{A } (g, \text{AM}\eta) \\
3 & \quad \vert\ \top_t & \top \text{ is a tautology} \\
4 & \quad \vert\ \Diamond \top_0 & 3{,}4\ \Diamond\text{I} \\
6 & \Diamond \top_0 & 3\text{-}5\ \text{AM}\eta \\
5 & \neg\Box\neg\top_0 & 6\ \text{MN}
\end{array}$$

$\Gamma' \vdash^*_{NL\alpha} \neg\Box\neg\top_0$; but by L3.7, Γ' is a $L\alpha$ scapegoat set; so there is a t such that $\Gamma' \vdash^*_{NL\alpha} 0.t$; so by construction, $\langle w_0, w_t\rangle \in R$ and η is satisfied.

(ρ) Suppose α includes condition ρ and $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^*_{NX\alpha} s.s$; so by construction, $\langle w_s, w_s\rangle \in R$ and ρ is satisfied.

(σ) Suppose α includes condition σ and $\langle w_s, w_t\rangle \in R$. If $w_s = w_t$ then σ is satisfied automatically. So suppose $w_s \neq w_t$; then by construction, $\Gamma' \vdash^*_{NX\alpha} s.t$; so by (AM$\sigma$), $\Gamma' \vdash^*_{NX\alpha} t.s$; so by construction, $\langle w_t, w_s\rangle \in R$ and σ is satisfied.

(τ) Suppose α includes condition τ and $\langle w_s, w_t\rangle, \langle w_t, w_u\rangle \in R$. If $w_s = w_t$ or $w_t = w_u$, then τ is satisfied automatically. So suppose $w_s \neq w_t$ and $w_t \neq w_u$; then by construction, $\Gamma' \vdash^*_{NX\alpha} s.t$ and $\Gamma' \vdash^*_{NX\alpha} t.u$; so by (AM$\tau$), $\Gamma' \vdash^*_{NX\alpha} s.u$; so by construction, $\langle w_s, w_u\rangle \in R$ and τ is satisfied.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L3.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

> Suppose Γ_0 is consistent and $A_0 \in \Gamma_0$; then by construction, $A_0 \in \Gamma'$; so $\Gamma' \vdash^*_{NX\alpha} A_0$; so since Γ_0 is consistent, by L3.8, $v_{w_0}(A) = 1$. And similarly for any $A_0 \in \Gamma_0$. But $m(0) = w_0$; so $v_m(\Gamma_0) = 1$.

Main result: Suppose $\Gamma \vDash_{X\alpha} A$ but $\Gamma \nvdash_{NX\alpha} A$. Then $\Gamma_0 \vDash^*_{X\alpha} A_0$ but $\Gamma_0 \nvdash^*_{NX\alpha} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NX\alpha} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NX\alpha} A_0$; so $\Gamma_0 \nvdash^*_{NX\alpha} \neg\neg A_0$; so by L3.2, $\Gamma_0 \cup \{\neg A_0\}$ is consistent; so by L3.9 and L3.10, there is an $X\alpha$ interpretation $\langle W, N, R, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_0\}) = 1$; so $v_{m(0)}(\neg A) = 1$; so by TX($\neg$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VXα^*, $\Gamma_0 \nvDash^*_{X\alpha} A_0$. This is impossible; reject the assumption: if $\Gamma \vDash_{X\alpha} A$, then $\Gamma \vdash_{NX\alpha} A$.

4 Conditional Logics: *Cx* (ch. 5)

4.1 Language / Semantic Notions

LCx The vocabulary consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\Box$, $\Diamond$ and $>$. Each propositional parameter is a formula; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$, $\Box A$, $\Diamond A$ and $(A > B)$.

ICx Where $\Im$ is the set of all formulas in the language, an interpretation is $\langle W, \{R_A \mid A \in \Im\}, v\rangle$ where W is a set of worlds, and v assigns 0 or 1 to parameters at worlds. The middle term is a *set* of access relations: for any formula A, there is an access relation R_A which says which worlds are A-accessible from any w. Say $f_A(w) = \{x \in W \mid wR_Ax\}$, and $[A] = \{w \mid v_w(A) = 1\}$. Then, where x is empty or indicates some combination of the following constraints,

(1) $f_A(w) \subseteq [A]$

(2) If $w \in [A]$, then $w \in f_A(w)$

(3) If $[A] \neq \phi$, then $f_A(w) \neq \phi$

(4) If $f_A(w) \subseteq [B]$ and $f_B(w) \subseteq [A]$, then $f_A(w) = f_B(w)$

(5) If $f_A(w) \cap [B] \neq \phi$, then $f_{A\wedge B}(w) \subseteq f_A(w)$

(6) If $x \in f_A(w)$ and $y \in f_A(w)$, then $x = y$

(7) If $x \in [A]$, and $y \in f_A(x)$, then $x = y$

$\langle W, \{R_A \mid A \in \Im\}, v\rangle$ is a *Cx* interpretation when it meets the constraints from x. System C has none of the extra constraints; *C+* is C with constraints (1) - (2); *CS* is C with constraints (1) - (5); *C1* is C with constraints (1) - (5) and (7); *C2* is C with constraints (1) - (5) and (6).

TC For complex expressions,

- $(\neg)$ $v_w(\neg A) = 1$ if $v_w(A) = 0$, and 0 otherwise.
- $(\wedge)$ $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.
- $(\vee)$ $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.
- $(\supset)$ $v_w(A \supset B) = 1$ if $v_w(A) = 0$ or $v_w(B) = 1$, and 0 otherwise.
- $(\equiv)$ $v_w(A \equiv B) = 1$ if $v_w(A) = v_w(B)$, and 0 otherwise.
- $(\Diamond)_v$ $v_w(\Diamond A) = 1$ if some $x \in W$ has $v_x(A) = 1$, and 0 otherwise.
- $(\Box)_v$ $v_w(\Box A) = 1$ if all $x \in W$ have $v_x(A) = 1$, and 0 otherwise.
- $(>)$ $v_w(A > B) = 1$ iff all $x \in W$ such that wR_Ax have $v_x(B) = 1$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VCx $\Gamma \models_{Cx} A$ iff there is no *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

4.2 Natural Derivations: *NCx*

Derivation systems *NCx* take over $\neg$, $\supset$, $\wedge$, $\vee$, $\equiv$, $\Box$ and $\Diamond$ rules from *NKv*. Thus modal rules are,

$$\Box\mathbf{I}v \;\; \begin{array}{|l} \begin{array}{|l} \top_t \\ \hline \\ \\ P_t \end{array} \\ \Box P_s \end{array} \qquad\qquad \Diamond\mathbf{E}v \;\; \begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} P_t \\ \hline \\ \\ Q_u \end{array} \\ Q_u \end{array}$$

where t does not appear in any undischarged premise or assumption

where t does not appear in any undischarged premise or assumption and is not u

$$\Box\mathbf{E}v \;\; \begin{array}{|l} \Box P_s \\ \\ P_t \end{array} \qquad\qquad \Diamond\mathbf{I}v \;\; \begin{array}{|l} P_t \\ \\ \Diamond P_s \end{array}$$

For $>$, let there be new subscripted expressions of the sort $A_{s/t}$ – which intuitively say $w_sR_Aw_t$. Expressions of this sort do not interact with other formulas except as follows (and so do not interact with rules of *NKv*):

$$\textbf{>I}\ \begin{array}{|l} \begin{array}{|l} P_{s/t} \\ \hline \\ Q_t \end{array} \\ (P > Q)_s \end{array} \qquad\qquad \textbf{$\ngtr$E}\ \begin{array}{|l} \neg(P > Q)_s \\ \begin{array}{|l} P_{s/t} \\ \neg Q_t \\ \hline \\ R_u \end{array} \\ R_u \end{array}$$

where t does not appear in any undischarged premise or assumption

where t does not appear in any undischarged premise or assumption and is not u

$$\textbf{>E}\ \begin{array}{|l} (P > Q)_s \\ P_{s/t} \\ \\ Q_t \end{array} \qquad\qquad \textbf{$\ngtr$I}\ \begin{array}{|l} P_{s/t} \\ \neg Q_t \\ \\ \neg(P > Q)_s \end{array}$$

Corresponding to constraints (1) - (7) are AMp1, AMp2, AMs1, AMs2, AMs3, AMrs, and two forms of AMdl. For AMrs $\mathcal{A}_{(t)}$ is an expression of the sort Q_t, $Q_{t/v}$, $Q_{v/t}$ or $Q_{t/t}$ with a subscript t, and $\mathcal{A}_{(u)}$ is like $\mathcal{A}_{(t)}$ except that some instance(s) of t are replaced by u. And similarly for AMdl.

$$\textbf{AMp1}\ \begin{array}{|l} P_{s/t} \\ \\ P_t \end{array} \qquad \textbf{AMp2}\ \begin{array}{|l} P_t \\ \\ P_{t/t} \end{array} \qquad \textbf{AMs1}\ \begin{array}{|l} \Diamond P_s \\ \begin{array}{|l} P_{s/t} \\ \hline \\ Q_u \end{array} \\ Q_u \end{array} \qquad \textbf{AMs2}\ \begin{array}{|l} (P > Q)_s \\ (Q > P)_s \\ P_{s/t} \\ \\ Q_{s/t} \end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

$$\textbf{AMs3}\ \begin{array}{|l} \neg(P > \neg Q)_s \\ (P \wedge Q)_{s/t} \\ \\ P_{s/t} \end{array} \qquad \textbf{AMrs}\ \begin{array}{|l} P_{s/t} \\ P_{s/u} \\ \mathcal{A}_{(t)} \\ \\ \mathcal{A}_{(u)} \end{array} \qquad \textbf{AMdl}\ \begin{array}{|l} P_s \\ P_{s/t} \\ \mathcal{A}_{(t)} \\ \\ \mathcal{A}_{(s)} \end{array} \quad \begin{array}{|l} P_s \\ P_{s/t} \\ \mathcal{A}_{(s)} \\ \\ \mathcal{A}_{(t)} \end{array}$$

In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an assumption for $\Box$Iv, $\Diamond$Ev, >I, $\ngtr$E or AMs1. Intuitively there are *plus* rules, rules for the *sphere* conception, and rules for the Stalnaker and Lewis alternatives. *NC* includes just the rules of *NKv* plus >I, >E, $\ngtr$I and $\ngtr$E (but, as below, the latter two are derived). Then,

NC+ has the rules of *NC* plus AMp1, AMp2

NCS has the rules of *NC* plus AMp1, AMp2, AMs1, AMs2, AMs3

NC1 has the rules of *NC* plus AMp1, AMp2, AMs1, AMs2, AMs3, AMdl

NC2 has the rules of *NC* plus AMP1, AMP2, AMS1, AMS2, AMS3, AMRS

Derived rules carry over from $NK\alpha$. Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas each with subscript 0. Then,

NCX $\Gamma \vdash_{NCx} A$ iff there is an *NCx* derivation of A_0 from Γ_0.

Examples. As first examples, $\not>$I and $\not>$E are derived rules in *NC*, and so in any *NCx*.

$\not>$I

1	$P_{s/t}$	P
2	$\neg Q_t$	P
3	$\mid (P > Q)_s$	A (c, $\neg$I)
4	$\mid Q_t$	1,3 $>$E
5	$\mid \neg Q_t$	2 R
6	$\neg(P > Q)_s$	3-5 $\neg$I

$\not>$E

1	$\neg(P > Q)_s$	P
2	$\mid \neg R_u$	A (c, $\neg$E)
3	$\mid\mid P_{s/t}$	A (g, $>$I)
4	$\mid\mid\mid \neg Q_t$	A (c, $\neg$E)
	$\mid\mid\mid \vdots$	from 1,3,4
5	$\mid\mid\mid R_u$	as for $\not>$E
6	$\mid\mid\mid \neg R_u$	2 R
7	$\mid\mid Q_t$	4-6 $\neg$E
8	$\mid (P > Q)_s$	3-7 $>$I
9	$\mid \neg(P > Q)_s$	1 R
10	R_u	2-9 $\neg$E

As final examples, here is a case in *NCS* using AMS3 and then again in *NC2* but without appeal to AMS3 (so that AMS3 is not necessary in *NC2* for the result). This last case is a bit messy, but should nicely illustrate use of the rules.

$A > B, \neg(A > \neg C) \vdash_{NCS} (A \wedge C) > B$

1	$(A > B)_0$	P
2	$\neg(A > \neg C)_0$	P
3	$(A \wedge C)_{0/1}$	A (g, >I)
4	$A_{0/1}$	2,3 AMs3
5	B_1	1,4 >E
6	$[(A \wedge C) > B]_0$	3-5 >I

$A > B, \neg(A > \neg C) \vdash_{NC2} (A \wedge C) > B$

1	$(A > B)_0$	P
2	$\neg(A > \neg C)_0$	P
3	$A_{0/1}$	A (g, 2 $\not>$E)
4	$\neg\neg C_1$	
5	$(A \wedge C)_{0/2}$	A (g, >I)
6	$(A \wedge C)_{0/3}$	A (g, >I)
7	$(A \wedge C)_3$	6 AMp1
8	A_3	7 $\wedge$E
9	$[(A \wedge C) > A]_0$	6-8 >I
10	$A_{0/3}$	A (g, >I)
11	A_3	10 AMp1
12	$\neg\neg C_3$	3,10,4 AMrs
13	C_3	12 DN
14	$(A \wedge C)_3$	11,13 $\wedge$I
15	$[A > (A \wedge C)]_0$	10-14 >I
16	$A_{0/2}$	9,15,5 AMs2
17	B_2	1,16 >E
18	$[(A \wedge C) > B]_0$	5-17 >I
19	$[(A \wedge C) > B]_0$	2,3-18 $\not>$E

The derivation on the left is a simple application of AMs3. On the right, we go for the final goal by $\not>$E.[4] The real work is getting $A_{0/2}$ so that we can use >E with (1). And we go for this by getting the conditionals that feed into AMs2, given that we already have $(A \wedge C)_{0/2}$.

4.3 Soundness and Completeness

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, \{R_A \,|\, A \in \Im\}, v\rangle$, let m be a map from subscripts into W. Say $\langle W, \{R_A \,|\, A \in \Im\}, v\rangle_m$ is $\langle W, \{R_A \,|\, A \in \Im\}, v\rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, and for each $A_{s/t} \in \Gamma$, $m(t) \in f_A(m(s))$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VCx* $\Gamma \models^*_{Cx} A_s$ $[\![A_{s/t}]\!]$ iff there is no Cx interpretation $\langle W, \{R_A | A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![m(t) \notin f_A(m(s))]\!]$.

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[4] As, given strategies from [6, chapter 6], we would jump on $\vee$E, $\exists$E or $\Diamond$E when available.

NCX* $\Gamma \vdash^*_{NCx} A_s$ $[\![A_{s/t}]\!]$ iff there is an *NCx* derivation of A_s $[\![A_{s/t}]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $A_{s/t}$). This is obvious for NCX*. In the other case, there is a $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ and $w \in W$ that makes all the members of Γ_0 true and A_0 false just in case there *is* a world in W that makes the unsubscripted members of Γ true and A false. For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 4.1 *NCx is sound: If* $\Gamma \vdash_{NCx} A$ *then* $\Gamma \models_{Cx} A$.

L4.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Cx} P_s$ $[\![P_{s/t}]\!]$, then $\Gamma' \models^*_{Cx} P_s$ $[\![P_{s/t}]\!]$.

Reasoning parallel to that for L2.1 of $NK\alpha$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NCx} \mathcal{P}$ then $\Gamma \models^*_{Cx} \mathcal{P}$. Suppose $\Gamma \vdash^*_{NCx} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{Cx} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![A_{s/t}]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{A_{s/t}\}]\!]$; so for any $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $[\![m(t) \in f_A(m(s))]\!]$; so there is no $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $[\![m(t) \notin f_A(m(s))]\!]$. So by VCX*, $\Gamma_1 \models^*_{Cx} A_s$ $[\![A_{s/t}]\!]$, where this is just to say, $\Gamma_1 \models^*_{Cx} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^*_{Cx} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\supset$I, $\supset$E, $\wedge$I, $\wedge$E, $\neg$I, $\neg$E, $\vee$I, $\vee$E, $\equiv$I, $\equiv$E, $\Box$Iv, $\Box$Ev, $\Diamond$Iv, $\Diamond$Ev, $>$I, $>$E or, depending on the system, AMP1, AMP2, AMS1, AMS2, AMS3, AMRS or AMDL. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\supset$I)

($\supset$E) If $\mathcal{P}_k$ arises by $\supset$E, then the picture is like this,

$$\begin{array}{r|l} i & (A \supset B)_s \\ j & A_s \\ & \\ k & B_s \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is B_s. By assumption, $\Gamma_i \models^*_{Cx} (A \supset B)_s$ and $\Gamma_j \models^*_{Cx} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} (A \supset B)_s$ and $\Gamma_k \models^*_{Cx} A_s$. Suppose $\Gamma_k \not\models^*_{Cx} B_s$; then by VCx*, there is some Cx interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(A \supset B) = 1$ and $v_{m(s)}(A) = 1$; from the former, by TC($\supset$), $v_{m(s)}(A) = 0$ or $v_{m(s)}(B) = 1$; so $v_{m(s)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\wedge$I)

($\wedge$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

$$\begin{array}{r|l|l} & & \underline{A_s} \\ & & \\ i & & B_t \\ j & & \neg B_t \\ k & \multicolumn{2}{l}{\neg A_s} \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{Cx} B_t$ and $\Gamma_j \models^*_{Cx} \neg B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L4.1, $\Gamma_k \cup \{A_s\} \models^*_{Cx} B_t$ and $\Gamma_k \cup \{A_s\} \models^*_{Cx} \neg B_t$. Suppose $\Gamma_k \not\models^*_{Cx} \neg A_s$; then by VCx*, there is a Cx interpretation $\langle W, \{R_A | A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TC($\neg$), $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by VCx*, $v_{m(t)}(B) = 1$ and $v_{m(t)}(\neg B) = 1$; from the latter, by TC($\neg$), $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} \neg A_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\neg$E)

($\vee$I)

($\vee$E)

($\equiv$I)

($\equiv$E)

($\Box$Iv) If $\mathcal{P}_k$ arises by $\Box$Iv, then the picture is like this,

$$\begin{array}{r|l|l}
 & & \top_t \\ \cline{3-3}
 & & \\
i & & A_t \\
k & \multicolumn{2}{l}{\Box A_s}
\end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\Box A_s$. By assumption, $\Gamma_i \models^*_{Cx} A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{\top_t\}$; so by L4.1, $\Gamma_k \cup \{\top_t\} \models^*_{Cx} A_t$. Suppose $\Gamma_k \not\models^*_{Cx} \Box A_s$; then by VCx*, there is a Cx interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\Box A) = 0$; so by TC($\Box$)$_v$, there is some $w \in W$ such that $v_w(A) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, \{R_A \mid A \in \Im\}, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and, as at any world, $v_{m'(t)}(\top) = 1$; so $v_{m'}(\Gamma_k \cup \{\top_t\}) = 1$; so by VCx*, $v_{m'(t)}(A) = 1$. But $m'(t) = w$; so $v_w(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} \Box A_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\Box$Ev) If $\mathcal{P}_k$ arises by $\Box$Ev, then the picture is like this,

$$\begin{array}{r|l}
i & \Box A_s \\
 & \\
k & A_t
\end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{Cx} \Box A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} \Box A_s$. Suppose $\Gamma_k \not\models^*_{Cx} A_t$; then by VCx*, there is some Cx interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(\Box A) = 1$; so by TC($\Box$)$_v$, any w has $v_w(A) = 1$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} A_t$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

($\Diamond$Iv)

($\Diamond$Ev) If $\mathcal{P}_k$ arises by $\Diamond$Ev, then the picture is like this,

$$\begin{array}{r|l|l}
i & \multicolumn{2}{l}{\Diamond A_s} \\
 & & A_t \\ \cline{3-3}
 & & \\
j & & B_u \\
k & \multicolumn{2}{l}{B_u}
\end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_i \models^*_{Cx} \Diamond A_s$ and $\Gamma_j \models^*_{Cx} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t\}$; so by L4.1, $\Gamma_k \models^*_{Cx} \Diamond A_s$ and $\Gamma_k \cup \{A_t\} \models^*_{Cx} B_u$. Suppose $\Gamma_k \not\models^*_{Cx} B_u$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$. Since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(\Diamond A) = 1$; so by TC$(\Diamond)_v$, there is some $w \in W$ such that $v_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, \{R_A \mid A \in \Im\}, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$, $v_{m'(t)}(A) = 1$; so $v_{m'}(\Gamma_k \cup \{A_t\}) = 1$; so by VCx*, $v_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_u$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(>I) If $\mathcal{P}_k$ arises by >I, then the picture is like this,

$$\begin{array}{r|l} & \vert\; \underline{A_{s/t}} \\ & \vert \\ i & \vert\; B_t \\ k & (A > B)_s \end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $(A > B)_s$. By assumption, $\Gamma_i \models^*_{Cx} B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_{s/t}\}$; so by L4.1, $\Gamma_k \cup \{A_{s/t}\} \models^*_{Cx} B_t$. Suppose $\Gamma_k \not\models^*_{Cx} (A > B)_s$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(A > B) = 0$; so by TC(>), there is some $w \in W$ such that $m(s)R_A w$ but $v_w(B) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, \{R_A \mid A \in \Im\}, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R_A$; so $v_{m'}(\Gamma_k \cup \{A_{s/t}\}) = 1$; so by VCx*, $v_{m'(t)}(B) = 1$. But $m'(t) = w$; so $v_w(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} (A > B)_s$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(>E) If $\mathcal{P}_k$ arises by >E, then the picture is like this,

$$\begin{array}{r|l} i & (A > B)_s \\ j & A_{s/t} \\ & \\ k & B_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is B_t. By assumption, $\Gamma_i \models^*_{Cx} (A > B)_s$ and

$\Gamma_j \models^*_{Cx} A_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} (A > B)_s$ and $\Gamma_k \models^*_{Cx} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{Cx} B_t$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(A > B) = 1$ and $\langle m(s), m(t)\rangle \in R_A$; from the former, by TC(>), any $w \in W$ such that $m(s)R_A w$ has $v_w(B) = 1$; so $v_{m(t)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_t$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMp1) If $\mathcal{P}_k$ arises by AMp1, then the picture is like this,

$$\begin{array}{r|l} i & A_{s/t} \\ & \\ k & A_t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A_t. Where this rule is in *NCx*, *Cx* includes condition (1). By assumption, $\Gamma_i \models^*_{Cx} A_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{Cx} A_t$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*, $m(t) \in f_A(m(s))$; so by condition (1), $m(t) \in [A]$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} A_t$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMp2) If $\mathcal{P}_k$ arises by AMp2, then the picture is like this,

$$\begin{array}{r|l} i & A_t \\ & \\ k & A_{t/t} \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $A_{t/t}$. Where this rule is in *NCx*, *Cx* includes condition (2). By assumption, $\Gamma_i \models^*_{Cx} A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} A_t$. Suppose $\Gamma_k \not\models^*_{Cx} A_{t/t}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $m(t) \notin f_A(m(t))$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(t)}(A) = 1$; so $m(t) \in [A]$; so by condition (2), $m(t) \in f_A(m(t))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} A_{t/t}$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMs1) If $\mathcal{P}_k$ arises by AMs1, then the picture is like this,

$$\begin{array}{r|l|l}
i & \Diamond A_s & \\
 & & A_{s/t} \\ \cline{3-3}
 & & \\
j & & B_u \\
k & B_u &
\end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is B_u. Where this rule is in *NCx*, *Cx* includes condition (3). By assumption, $\Gamma_i \models^*_{Cx} \Diamond A_s$ and $\Gamma_j \models^*_{Cx} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_{s/t}\}$; so by L4.1, $\Gamma_k \models^*_{Cx} \Diamond A_s$ and $\Gamma_k \cup \{A_{s/t}\} \models^*_{Cx} B_u$. Suppose $\Gamma_k \not\models^*_{Cx} B_u$; then by VCx*, there is a *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$. Since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(\Diamond A) = 1$; so by TC$(\Diamond)_v$, there is some $w \in W$ such that $v_w(A) = 1$; so $w \in [A]$ and $[A] \neq \phi$; so by condition (3), $f_A(m(s)) \neq \phi$; so there is some $x \in f_A(m(s))$. Now consider a map m' like m except that $m'(t) = x$, and consider $\langle W, \{R_A \mid A \in \Im\}, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = x$ and $m'(s) = m(s)$, $m'(t) \in f_A(m'(s))$; so $v_{m'}(\Gamma_k) = 1$ and $\langle m'(s), m'(t)\rangle \in R_A$; so $v_{m'}(\Gamma_k \cup \{A_{s/t}\}) = 1$; so by VCx*, $v_{m'(u)}(B) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_u$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$.

(AMs2) If $\mathcal{P}_k$ arises by AMs2, then the picture is like this,

$$\begin{array}{r|l}
h & (A > B)_s \\
i & (B > A)_s \\
j & A_{s/t} \\
 & \\
k & B_{s/t}
\end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $B_{s/t}$. Where this rule is in *NCx*, *Cx* includes condition (4). By assumption, $\Gamma_h \models^*_{Cx} (A > B)_s$, $\Gamma_i \models^*_{Cx} (B > A)_s$ and $\Gamma_j \models^*_{Cx} A_{s/t}$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} (A > B)_s$, $\Gamma_k \models^*_{Cx} (B > A)_s$, and $\Gamma_k \models^*_{Cx} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{Cx} B_{s/t}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $m(t) \notin f_B(m(s))$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(A > B) = 1$, $v_{m(s)}(B > A) = 1$; and $m(t) \in f_A(m(s))$. Suppose $w \in f_A(m(s))$; then $m(s)R_A w$ and since $v_{m(s)}(A > B) = 1$, by TC(>), $v_w(B) = 1$; so $w \in [B]$ and, generalizing, we have that $f_A(m(s)) \subseteq$

$[B]$. Suppose $w \in f_B(m(s))$; then $m(s)R_Bw$ and since $v_{m(s)}(B > A) = 1$, by TC(>), $v_w(A) = 1$; so $w \in [A]$ and, generalizing, we have that $f_B(m(s)) \subseteq [A]$. So $f_A(m(s)) \subseteq [B]$ and $f_B(m(s)) \subseteq [A]$; so by condition (4), $f_A(m(s)) = f_B(m(s))$; thus since $m(t) \in f_A(m(s))$, $m(t) \in f_B(m(s))$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{Cx} B_{s/t}$, which is to say, $\Gamma_k \vDash^*_{Cx} \mathcal{P}_k$.

(AMs3) If $\mathcal{P}_k$ arises by AMs3, then the picture is like this,

$$\begin{array}{r|l} i & \neg(A > \neg B)_s \\ j & (A \wedge B)_{s/t} \\ & \\ k & A_{s/t} \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $A_{s/t}$. Where this rule is in *NCx*, *Cx* includes condition (5). By assumption, $\Gamma_i \vDash^*_{Cx} \neg(A > \neg B)_s$ and $\Gamma_j \vDash^*_{Cx} (A \wedge B)_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \vDash^*_{Cx} \neg(A > \neg B)_s$, and $\Gamma_k \vDash^*_{Cx} (A \wedge B)_{s/t}$. Suppose $\Gamma_k \nvDash^*_{Cx} A_{s/t}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $m(t) \notin f_A(m(s))$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(\neg(A > \neg B)) = 1$, and $m(t) \in f_{A \wedge B}(m(s))$. Since $v_{m(s)}(\neg(A > \neg B)) = 1$, by TC(¬), $v_m(s)(A > \neg B) = 0$; so by TC(>), there is some $w \in W$ such that $m(s)R_Aw$ and $v_w(\neg B) = 0$; so by TC(¬), $v_w(B) = 1$; but $w \in f_A(m(s))$; so $f_A(m(s)) \cap [B] \neq \phi$; so by condition (5), $f_{A \wedge B}(m(s)) \subseteq f_A(m(s))$; so $m(t) \in f_A(m(s))$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{Cx} A_{s/t}$, which is to say, $\Gamma_k \vDash^*_{Cx} \mathcal{P}_k$.

(AMrs) If $\mathcal{P}_k$ arises by AMrs, then the picture is like this,

$$\begin{array}{r|l} h & A_{s/t} \\ i & A_{s/u} \\ j & \mathcal{Q}_{(t)} \\ & \\ k & \mathcal{Q}_{(u)} \end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $\mathcal{Q}_{(u)}$. Suppose $\mathcal{Q}_{(t)}$ is some B_t and $\mathcal{Q}_{(u)}$ is B_u. Where this rule is in *NCx*, *Cx* includes condition (6). By assumption, $\Gamma_h \vDash^*_{Cx} A_{s/t}$, $\Gamma_i \vDash^*_{Cx} A_{s/u}$ and $\Gamma_j \vDash^*_{Cx} B_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \vDash^*_{Cx} A_{s/t}$, $\Gamma_k \vDash^*_{Cx} A_{s/u}$, and $\Gamma_k \vDash^*_{Cx} B_t$. Suppose $\Gamma_k \nvDash^*_{Cx} B_u$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VCx*,

$m(t) \in f_A(m(s))$, $m(u) \in f_A(m(s))$, and $v_{m(t)}(B) = 1$. With the first two of these, by condition (6), $m(t) = m(u)$; so $v_{m(u)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \not\models^*_{Cx} B_u$, which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$. And similarly when $\mathcal{Q}_{(t)}$ is $B_{t/v}$, $B_{v/t}$, or $B_{t/t}$.

(AMdl) If $\mathcal{P}_k$ arises by AMdl, then the picture is like this,

$$\begin{array}{r|l} h & A_s \\ i & A_{s/t} \\ j & \mathcal{Q}_{(t)} \\ & \\ k & \mathcal{Q}_{(s)} \end{array} \qquad \text{or} \qquad \begin{array}{r|l} h & A_s \\ i & A_{s/t} \\ j & \mathcal{Q}_{(s)} \\ & \\ k & \mathcal{Q}_{(t)} \end{array}$$

where $h, i, j < k$ and, in the left-hand case, $\mathcal{P}_k$ is $\mathcal{Q}_{(s)}$. Suppose $\mathcal{Q}_{(t)}$ is of the sort $B_{t/v}$ and $\mathcal{Q}_{(s)}$ is $B_{s/v}$. Where this rule is in *NCx*, *Cx* includes condition (7). By assumption, $\Gamma_h \models^*_{Cx} A_s$, $\Gamma_i \models^*_{Cx} A_{s/t}$ and $\Gamma_j \models^*_{Cx} B_{t/v}$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L4.1, $\Gamma_k \models^*_{Cx} A_s$, $\Gamma_k \models^*_{Cx} A_{s/t}$, and $\Gamma_k \models^*_{Cx} B_{t/v}$. Suppose $\Gamma_k \not\models^*_{Cx} B_{s/v}$; then by VCx*, there is some *Cx* interpretation $\langle W, \{R_A \mid A \in \Im\}, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(v)\rangle \notin R_B$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(s)}(A) = 1$, $m(t) \in f_A(m(s))$, and $\langle m(t), m(v)\rangle \in R_B$. From the first of these, $m(s) \in [A]$; so by condition (7), $m(s) = m(t)$; so $\langle m(s), m(v)\rangle \in R_B$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Cx} B_{s/v}$ which is to say, $\Gamma_k \models^*_{Cx} \mathcal{P}_k$. And similarly when $\mathcal{Q}_{(t)}$ is B_t, $B_{v/t}$ or $B_{t/t}$. And similarly in the right-hand case.

For any i, $\Gamma_i \models^*_{Cx} \mathcal{P}_i$.

THEOREM 4.2 *NCx is complete: if* $\Gamma \models_{Cx} A$ *then* $\Gamma \vdash_{NCx} A$.

Suppose $\Gamma \models_{Cx} A$; then $\Gamma_0 \models^*_{Cx} A_0$; we show that $\Gamma_0 \vdash^*_{NCx} A_0$. Again, this reduces to the standard notion. For the following, fix on some particular constraint(s) x. Then definitions of *consistency* etc. are relative to it.

Con Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NCx} A_s$ and $\Gamma \vdash^*_{NCx} \neg A_s$.

L4.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NCx} \neg P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Reasoning parallel to L2.2 for $NK\alpha$.

L4.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$

Proof by construction as for L2.3 for $NK\alpha$.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{\mathit{NCx}} A_s$ or $\Gamma \vdash^*_{\mathit{NCx}} \neg A_s$.

SGT Γ is a SCAPEGOAT set for $\Box$ iff for every formula of the form $\neg\Box A_s$, if $\Gamma \vdash^*_{\mathit{NCx}} \neg\Box A_s$ then there is some t such that $\Gamma \vdash^*_{\mathit{NCx}} \neg A_t$.

Γ is a SCAPEGOAT set for $>$ iff for any formula of the form $\neg(A > B)_s$, if $\Gamma \vdash^*_{\mathit{NCx}} \neg(A > B)_s$ then there is some t such that $\Gamma \vdash^*_{\mathit{NCx}} A_{s/t}$ and $\Gamma \vdash^*_{\mathit{NCx}} \neg B_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L4.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

$$\begin{array}{ll} \Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \vdash^*_{\mathit{NCx}} \neg A_s \\ \Omega_{i^*} = \Omega_{i-1} \cup \{A_s\} & \text{if } \Omega_{i-1} \nvdash^*_{\mathit{NCx}} \neg A_s \end{array}$$

and

$$\begin{array}{ll} \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } \neg\Box P_s \text{ or } \neg(P > Q)_s \\ \Omega_i = \Omega_{i^*} \cup \{\neg P_t\} & \text{if } A_s \text{ is of the form } \neg\Box P_s \\ \Omega_i = \Omega_{i^*} \cup \{P_{s/t}, \neg Q_t\} & \text{if } A_s \text{ is of the form } \neg(P > Q)_s \end{array}$$

-where t is the first subscript not included in Ω_{i^*}

then

$$\Gamma' = \textstyle\bigcup_{i \geq 0} \Omega_i$$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L4.4 For any s included in Γ', Γ' is s-maximal.

Reasoning parallel to L2.4 for $NK\alpha$.

L4.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$, (iii) $\Omega_{k^*} \cup \{\neg P_t\}$ or (iv) $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NCx} \neg A_s$; so by L4.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{\neg P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg\Box P_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{\neg P_t\} \vdash^*_{NCx} A_u$ and $\Omega_{k^*} \cup \{\neg P_t\} \vdash^*_{NCx} \neg A_u$. So reason as follows,

$$\begin{array}{r|lll}
1 & \Omega_{k^*} & & \\
2 & \vert\ \underline{\top_t} & & \text{A } (g, \Box\text{I}\upsilon) \\
3 & \vert\ \vert\ \underline{\neg P_t} & & \text{A } (c, \neg\text{E}) \\
4 & \vert\ \vert\ A_u & & \text{from } \Omega_{k^*} \cup \{\neg P_t\} \\
5 & \vert\ \vert\ \neg A_u & & \text{from } \Omega_{k^*} \cup \{\neg P_t\} \\
6 & \vert\ P_t & & \text{3-5 } \neg\text{E} \\
7 & \Box P_s & & \text{2-6 } \Box\text{I}\upsilon
\end{array}$$

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NCx} \Box P_s$; but $\neg\Box P_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NCx} \neg\Box P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg(P > Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\neg A_u$ such that $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\} \vdash^*_{NCx} A_u$ and $\Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\} \vdash^*_{NCx} \neg A_u$. So reason as follows,

$$\begin{array}{r|lll}
1 & \Omega_{k^*} & & \\
2 & \vert\ \underline{P_{s/t}} & & \text{A } (g, >\text{I}) \\
3 & \vert\ \vert\ \underline{\neg Q_t} & & \text{A } (c, \neg\text{E}) \\
4 & \vert\ \vert\ A_u & & \text{from } \Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\} \\
5 & \vert\ \vert\ \neg A_u & & \text{from } \Omega_{k^*} \cup \{P_{s/t}, \neg Q_t\} \\
6 & \vert\ Q_t & & \text{3-5 } \neg\text{E} \\
7 & (P > Q)_s & & \text{2-6 } >\text{I}
\end{array}$$

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NCx} (P > Q)_s$; but $\neg(P > Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NCx} \neg(P > Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L4.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 for $NK\alpha$.

L4.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $\Box$ and $>$.

For $\Box$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NCx} \neg\Box P_s$. By L4.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NCx} \neg\neg\Box P_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg\Box P_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{\neg P_t\}$; so by construction, $\neg P_t \in \Gamma'$; so $\Gamma' \vdash^*_{NCx} \neg P_t$. So Γ' is a scapegoat set for $\Box$.

For $>$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NCx} \neg(P > Q)_s$. By L4.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NCx} \neg\neg(P > Q)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg(P > Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_{s/t}, \neg Q_t\}$; so by construction, $P_{s/t} \in \Gamma'$ and $\neg Q_t \in \Gamma'$; so $\Gamma' \vdash^*_{NCx} P_{s/t}$ and $\Gamma' \vdash^*_{NCx} \neg Q_t$. So Γ' is a scapegoat set for $>$.

C(I) We construct an interpretation $I = \langle W, \{R_A \mid A \in \Im\}, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ', except that in *C1*, if there is some A such that $\Gamma' \vdash^*_{NC1} A_s$ and $\Gamma' \vdash^*_{NC1} A_{s/t}$ then $w_s = w_t$, and in *C2*, if there is some A such that $\Gamma' \vdash^*_{NC2} A_{s/t}$ and $\Gamma' \vdash^*_{NC2} A_{s/u}$ then $w_t = w_u$ (we could do this, in the usual way, by establishing equivalence classes from members of W). Then $\langle w_s, w_t\rangle \in R_A$ iff $\Gamma' \vdash^*_{NCx} A_{s/t}$; and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NCx} p_s$.

Note that the specification is consistent for *C1* and *C2*: Say $\mathcal{P}_{(s)}$ is some p_s, $P_{s/v}$, $P_{v/s}$ or $P_{s/s}$. (i) Suppose $w_s = w_t$ and $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(s)}$. Since $w_s = w_t$ there is some A such that $\Gamma' \vdash^*_{NC1} A_s$ and $\Gamma' \vdash^*_{NC1} A_{s/t}$; so by AMDL, $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(t)}$. And similarly if $w_s = w_t$ and $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(t)}$, then $\Gamma' \vdash^*_{NC1} \mathcal{P}_{(s)}$. (ii) Suppose $w_t = w_u$ and $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(t)}$. Since $w_t = w_u$, there is some A such that $\Gamma' \vdash^*_{NC2} A_{s/t}$ and $\Gamma' \vdash^*_{NC2} A_{s/u}$; so by AMRS, $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(u)}$. And similarly if $w_t = w_u$ and $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(u)}$, then $\Gamma' \vdash^*_{NC2} \mathcal{P}_{(t)}$.

L4.8 If Γ_0 is consistent then for $\langle W, \{R_A \mid A \in \Im\}, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L4.4, Γ' is s-maximal. By L4.6 and L4.7, Γ' is consistent and a scapegoat set for $\Box$ and $>$. Now by induction on the number of operators in A_s, *Basis:* If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NCx} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

Assp: For any i, $0 \leq i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

If A_s has k operators, then it is of the form $\neg P_s$, $(P \supset Q)_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \equiv Q)_s$, $\Box P_s$, $\Diamond P_s$ or $(P > Q)_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TC($\neg$), $v_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NCx} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NCx} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NCx} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NCx} A_s$; then $\Gamma' \vdash^*_{NCx} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NCx} P_s$; so by assumption, $v_{w_s}(P) = 0$; so by TC($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

($\supset$)

($\wedge$)

($\vee$)

($\equiv$)

($\Box$) A_s is $\Box P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NCx} A_s$; then $v_{w_s}(\Box P) = 1$ but $\Gamma' \nvdash^*_{NCx} \Box P_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NCx} \neg\Box P_s$; so, since Γ' is a scapegoat set for $\Box$, there is some t such that $\Gamma' \vdash^*_{NCx} \neg P_t$; so by consistency, $\Gamma' \nvdash^*_{NCx} P_t$; so by assumption, $v_{w_t}(P) = 0$; so by TC($\Box$)$_\upsilon$, $v_{w_s}(\Box P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NCx} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NCx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NCx} \Box P_s$ but $v_{w_s}(\Box P) = 0$. From the the latter, by TC($\Box$)$_\upsilon$, there is some $w_t \in W$ such that $v_{w_t}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NCx} P_t$; but since $w_t \in W$, by construction, t appears in Γ' so by ($\Box$Eυ), $\Gamma' \vdash^*_{NCx} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NCx} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

($\Diamond$)

($>$) A_s is $(P > Q)_s$. Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NCx} A_s$; then $v_{w_s}(P > Q) = 1$ but $\Gamma' \nvdash^*_{NCx} (P > Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NCx} \neg(P > Q)_s$; so, since Γ' is a scapegoat set for $>$, there is some t such that $\Gamma' \vdash^*_{NCx} P_{s/t}$ and $\Gamma' \vdash^*_{NCx} \neg Q_t$; from the first, by construction, $\langle w_s, w_t\rangle \in R_P$; and from the second, by consistency, $\Gamma' \nvdash^*_{NCx} Q_t$; so by assumption, $v_{w_t}(Q) = 0$; so by TC($>$), $v_{w_s}(P > Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NCx} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NCx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NCx} (P > Q)_s$ but $v_{w_s}(P > Q) = 0$. From the the latter, by TC($>$), there is some $w_t \in W$ such that $\langle w_s, w_t\rangle \in R_P$ and $v_{w_t}(Q) = 0$; from the first of these, by construction, $\Gamma' \vdash^*_{NCx} P_{s/t}$; and from the second, by assumption, $\Gamma' \nvdash^*_{NCx} Q_t$; but by ($>$E), $\Gamma' \vdash^*_{NCx} Q_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NCx} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NCx} A_s$.

L4.9 If Γ_0 is consistent, then $\langle W, \{R_A \mid A \in \Im\}, v\rangle$ constructed as above is a *Cx* interpretation.

In each case, we need to show that the interpretation meets the condition(s) x. Suppose Γ_0 is consistent.

(1) If (1) is in *Cx*, then AMP1 is in *NCx*. Suppose $w_t \in f_A(w_s)$; then $\langle w_s, w_t\rangle \in R_A$; so by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$; so by AMP1, $\Gamma' \vdash^*_{NCx} A_t$; so by L4.8, $v_{w_t}(A) = 1$; so $w_t \in [A]$. So $f_A(w_s) \subseteq [A]$.

(2) If (2) is in *Cx* then AMP2 is in *NCx*. Suppose $w_s \in [A]$; then $v_{w_s}(A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} A_s$; so by AMP2, $\Gamma' \vdash^*_{NCx} A_{s/s}$; so by construction, $\langle w_s, w_s\rangle \in R_A$; so $w_s \in f_A(w_s)$.

(3) If (3) is in *Cx* then AMS1 is in *NCx*. Suppose $[A] \neq \phi$ but $f_A(w_s) = \phi$. From the former, there is some $w_t \in W$ such that $v_{w_t}(A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} A_t$; so by ($\Diamond$I$v$), $\Gamma' \vdash^*_{NCx} \Diamond A_s$. From the latter, there is no w_u such that $w_s R_A w_u$; so there is no w_u such that $w_s R_A w_u$ and $v_{w_u}(B) = 0$, and there is no w_u such that $w_s R_A w_u$ and $v_{w_u}(\neg B) = 0$; so by TC($>$), $v_{w_s}(A > B) = 1$ and $v_{w_s}(A > \neg B) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} (A > B)_s$ and $\Gamma' \vdash^*_{NCx} (A > \neg B)_s$. So reason as follows,

$$\begin{array}{r|lll}
1 & \Gamma' & & \\
2 & \Diamond A_s & & \text{from } \Gamma' \\
3 & (A > B)_s & & \text{from } \Gamma' \\
4 & (A > \neg B)_s & & \text{from } \Gamma' \\
5 & \quad \vert\ \underline{A_{s/t}} & & \text{A } (g, 2 \text{ AMs1}) \\
6 & \quad \vert \quad \vert\ \underline{\Diamond A_s} & & \text{A } (c, \neg\text{I}) \\
7 & \quad \vert \quad \vert\ B_t & & 3{,}5 >\text{E} \\
8 & \quad \vert \quad \vert\ \neg B_t & & 4{,}5 >\text{E} \\
9 & \quad \vert\ \neg\Diamond A_s & & 6\text{-}8\ \neg\text{I} \\
10 & \neg\Diamond A_s & & 2{,}5\text{-}9 \text{ AMs1}
\end{array}$$

So $\Gamma' \vdash^*_{NCx} \neg\Diamond A_s$; and since by L4.6, Γ' is consistent, $\Gamma' \nvdash^*_{NCx} \Diamond A_s$. This is impossible; reject the assumption: if $[A] \neq \phi$, then $f_A(w_s) \neq \phi$.

(4) If (4) is in *Cx* then AMs2 is in *NCx*. Suppose $f_A(w_s) \subseteq [B]$ and $f_B(w_s) \subseteq [A]$. Then any $x \in W$ such that $w_s R_A x$ has $v_x(B) = 1$ and any $y \in W$ such that $w_s R_B y$ has $v_y(A) = 1$; so by TC(>), $v_{w_s}(A > B) = 1$ and $v_{w_s}(B > A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} (A > B)_s$ and $\Gamma' \vdash^*_{NCx} (B > A)_s$. Suppose $w_t \in f_A(w_s)$; then by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$; so by AMs2, $\Gamma' \vdash^*_{NCx} B_{s/t}$; so by construction, $w_t \in f_B(w_s)$. Suppose $w_t \in f_B(w_s)$; then by construction, $\Gamma' \vdash^*_{NCx} B_{s/t}$; so by AMs2, $\Gamma' \vdash^*_{NCx} A_{s/t}$; so by construction, $w_t \in f_A(w_s)$. So $f_A(w_s) = f_B(w_s)$.

(5) If (5) is in *Cx* then AMs3 is in *NCx*. Suppose $f_A(w_s) \cap [B] \neq \phi$ but $f_{A\wedge B}(w_s) \not\subseteq f_A(w_s)$. From the former, there is some $w_t \in f_A(w_s)$ such that $v_{w_t}(B) = 1$; so by TC(¬), $v_{w_t}(\neg B) = 0$; so by TC(>), $v_{w_s}(A > \neg B) = 0$; so by TC(¬), $v_{w_s}(\neg(A > \neg B)) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} \neg(A > \neg B)_s$. From the latter, there is some w_u such that $w_u \in f_{A\wedge B}(w_s)$ but $w_u \notin f_A(w_s)$. From the first of these, by construction, $\Gamma' \vdash^*_{NCx} (A \wedge B)_{s/u}$; so by AMs3, $\Gamma' \vdash^*_{NCx} A_{s/u}$; so by construction, $w_u \in f_A(w_s)$. This is impossible; reject the assumption: if $f_A(w_s) \cap [B] \neq \phi$ then $f_{A\wedge B}(w_s) \subseteq f_A(w_s)$.

(6) Suppose (6) is in *Cx*, $w_t \in f_A(w_s)$ and $w_u \in f_A(w_s)$. Then by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$ and $\Gamma' \vdash^*_{NCx} A_{s/u}$; and by construction, since we are in *C2*, $w_t = w_u$.

(7) Suppose (7) is in *Cx*, $w_s \in [A]$ and $w_t \in f_A(w_s)$. Since $w_s \in [A]$, $v_{w_s}(A) = 1$; so by L4.8, $\Gamma' \vdash^*_{NCx} A_s$; and since $w_t \in f_A(w_s)$, by construction, $\Gamma' \vdash^*_{NCx} A_{s/t}$. So by construction, since we are in *C1*, $w_s = w_t$.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L4.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 for $NK\alpha$.

Main result: Suppose $\Gamma \models_{Cx} A$ but $\Gamma \nvdash_{NCx} A$. Then $\Gamma_0 \models^*_{Cx} A_0$ but $\Gamma_0 \nvdash^*_{NCx} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NCx} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NCx} A_0$; so $\Gamma_0 \nvdash^*_{NCx} \neg\neg A_0$; so by L4.2, $\Gamma_0 \cup \{\neg A_0\}$ is consistent; so by L4.9 and L4.10, there is a Cx interpretation $\langle W, \{R_A | A \in \Im\}, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_0\}) = 1$; so $v_{m(0)}(\neg A) = 1$; so by TC($\neg$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VCx*, $\Gamma_0 \not\models^*_{Cx} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{Cx} A$, then $\Gamma \vdash_{NCx} A$.

5 Intuitionistic Logic: *IL* (ch. 6)

5.1 Language / Semantic Notions

LIL The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\wedge$, $\vee$, $\neg$, and $\sqsupset$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $(A \wedge B)$, $(A \vee B)$, $\neg A$, and $(A \sqsupset B)$.

IIL An INTERPRETATION is a triple $\langle W, R, v\rangle$ where W is a set of worlds, R is a subset of $W^2 = W \times W$, and v is a function such that for any $w \in W$ and p, $v_w(p) = 1$ or $v_w(p) = 0$. For $x, y, z \in W$, an interpretation is subject to the conditions,

ρ	for all x, xRx	reflexivity
τ	for all x, y, z, if xRy and yRz then xRz	transitivity
h	for any parameter p, if $v_x(p) = 1$, and xRy, then $v_y(p) = 1$	heredity

We think of worlds as representing a state of information at a given time. $v_w(p) = 1$ when p is proved at state w. The heredity condition guarantees that what is proved at one stage remains proved at the next. Notice that $v_w(p) = 0$ does not indicate that p is *false* – but rather that p *isn't proved*.

TIL For complex expressions,

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\rightharpoondown$) $v_w(\rightharpoondown A) = 1$ if all $x \in W$ such that wRx have $v_x(A) = 0$, and 0 otherwise.

($\sqsupset$) $v_w(A \sqsupset B) = 1$ if all $x \in W$ such that wRx have either $v_x(A) = 0$ or $v_x(B) = 1$, and 0 otherwise.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VIL $\Gamma \models_{IL} A$ iff there is no *IL* interpretation $\langle W, R, v\rangle$ and $w \in W$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

5.2 Natural Derivations: *NIL*

Augment the language for intuionistic logic to include expressions with subscripts and expressions of the sort $s.t$ as for $NK\alpha$, along with a unary operator, $\sim$. Intuitively, $\sim A$ indicates that A is not (yet) proven. There is one new rule for the heredity condition. Otherwise, rules are as in $NK\rho\tau$ with $\sim$ like $\neg$, and rules for $\sqsupset$ and $\rightharpoondown$ on the analogy of $\mathrel{-\!\!3}$ and $\Box\neg$.

$$\mathbf{R}\ \begin{array}{|l} P_s \\ \\ P_s \end{array} \qquad \mathbf{H}\ \begin{array}{|l} P_s \\ s.t \\ \\ P_t \end{array}$$

where P includes no instance of $\sim$

$$\wedge\mathbf{I}\ \begin{array}{|l} P_s \\ Q_s \\ \\ (P \wedge Q)_s \end{array} \qquad \wedge\mathbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ P_s \end{array} \qquad \wedge\mathbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ Q_s \end{array}$$

$$\vee\mathbf{I}\ \begin{array}{|l} P_s \\ \\ (P \vee Q)_s \end{array} \qquad \vee\mathbf{I}\ \begin{array}{|l} P_s \\ \\ (Q \vee P)_s \end{array} \qquad \vee\mathbf{E}\ \begin{array}{|l} (P \vee Q)_s \\ \begin{array}{|l} P_s \\ \hline \\ R_t \end{array} \\ \begin{array}{|l} Q_s \\ \hline \\ R_t \end{array} \\ R_t \end{array}$$

$$\sim\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} P_s \\ \hline \\ Q_t \\ \sim Q_t \end{array} \\ \sim P_s \end{array} \qquad \sim\mathbf{E}\ \begin{array}{|l} \begin{array}{|l} \sim P_s \\ \hline \\ Q_t \\ \sim Q_t \end{array} \\ P_s \end{array}$$

$$\sqsupset\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ P_t \\ \hline \\ Q_t \end{array} \\ (P \sqsupset Q)_s \end{array} \qquad \sqsupset\mathbf{E}\ \begin{array}{|l} (P \sqsupset Q)_s \\ s.t \\ P_t \\ \\ Q_t \end{array} \qquad \mathbf{AM}\rho\ \begin{array}{|l} \\ \\ s.s \end{array}$$

where t does not appear in any undischarged premise or assumption

$$\textbf{$\neg$I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \hline \\ \sim P_t \end{array} \\ \neg P_s \end{array} \qquad\qquad \textbf{$\neg$E}\ \begin{array}{|l} \neg P_s \\ s.t \\ \\ \sim P_t \end{array} \qquad\qquad \textbf{AM}\tau\ \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

where t does not appear in any undischarged premise or assumption

Every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for $\sqsupset$I or $\neg$I. Where the members of Γ and A are formulas in the original language for intuitionistic logic (without subscripts and without $\sim$), let let the members of Γ_0 be the formulas in Γ, each with subscript 0. Then,

NIL $\Gamma \vdash_{NIL} A$ iff there is an *NIL* derivation of A_0 from the members of Γ_0.

Examples. Here are instances of the more interesting standard axioms for intuitionistic logic. Note that our account of a derivation guarantees that $\sim$ is not an operator in any of A, B, or C.

A1 $\vdash_{NIL} A \sqsupset (B \sqsupset A)$

1	$\vert\ \vert\ 0.1$	A (g, $\sqsupset$I)
2	$\vert\ \vert\ \underline{A_1}$	
3	$\vert\ \vert\ \vert\ 1.2$	A (g, $\sqsupset$I)
4	$\vert\ \vert\ \vert\ \underline{B_2}$	
5	$\vert\ \vert\ \vert\ A_2$	2,3 H
6	$\vert\ \vert\ (B \sqsupset A)_1$	3-5 $\sqsupset$I
7	$\vert\ [A \sqsupset (B \sqsupset A)]_0$	1-6 $\sqsupset$I

A2 $\vdash_{NIL} (A \sqsupset B) \sqsupset [(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)]$

1	$\mid\ \mid\ 0.1$	A (g, $\sqsupset$I)
2	$\mid\ \mid\ (A \sqsupset B)_1$	
3	$\mid\ \mid\ \mid\ 1.2$	A (g, $\sqsupset$I)
4	$\mid\ \mid\ \mid\ (A \sqsupset (B \sqsupset C))_2$	
5	$\mid\ \mid\ \mid\ \mid\ 2.3$	A (g, $\sqsupset$I)
6	$\mid\ \mid\ \mid\ \mid\ A_3$	
7	$\mid\ \mid\ \mid\ \mid\ 1.3$	3,5 AMτ
8	$\mid\ \mid\ \mid\ \mid\ B_3$	2,7,6 $\sqsupset$E
9	$\mid\ \mid\ \mid\ \mid\ (B \sqsupset C)_3$	4,5,6 $\sqsupset$E
10	$\mid\ \mid\ \mid\ \mid\ 3.3$	AMρ
11	$\mid\ \mid\ \mid\ \mid\ C_3$	9,10,8 $\sqsupset$E
12	$\mid\ \mid\ \mid\ (A \sqsupset C)_2$	5-11 $\sqsupset$I
13	$\mid\ \mid\ [(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)]_1$	3-12 $\sqsupset$I
14	$\mid\ ((A \sqsupset B) \sqsupset [(A \sqsupset (B \sqsupset C)) \sqsupset (A \sqsupset C)])_0$	1-13 $\sqsupset$I

A3 $\vdash_{NIL} A \sqsupset (B \sqsupset (A \wedge B))$

A4 $\vdash_{NIL} (A \wedge B) \sqsupset A$

A5 $\vdash_{NIL} (A \wedge B) \sqsupset B$

A6 $\vdash_{NIL} A \sqsupset (A \vee B)$

A7 $\vdash_{NIL} B \sqsupset (A \vee B)$

A8 $\vdash_{NIL} (A \sqsupset C) \sqsupset [(B \sqsupset C) \sqsupset ((A \vee B) \sqsupset C)]$

A9 $\vdash_{NIL} (A \sqsupset B) \sqsupset [(A \sqsupset \neg B) \sqsupset \neg A]$

1	$\mid\ \mid\ 0.1$	A (g, $\sqsupset$I)
2	$\mid\ \mid\ (A \sqsupset B)_1$	
3	$\mid\ \mid\ \mid\ 1.2$	A (g, $\sqsupset$I)
4	$\mid\ \mid\ \mid\ (A \sqsupset \neg B)_2$	
5	$\mid\ \mid\ \mid\ \mid\ 2.3$	A (g, $\neg$I)
6	$\mid\ \mid\ \mid\ \mid\ \mid\ A_3$	A (c, $\sim$I)
7	$\mid\ \mid\ \mid\ \mid\ \mid\ 1.3$	3,5 AMτ
8	$\mid\ \mid\ \mid\ \mid\ \mid\ B_3$	2,7,6 $\sqsupset$E
9	$\mid\ \mid\ \mid\ \mid\ \mid\ \neg B_3$	4,5,6 $\sqsupset$E
10	$\mid\ \mid\ \mid\ \mid\ \mid\ 3.3$	AMρ
11	$\mid\ \mid\ \mid\ \mid\ \mid\ {\sim} B_3$	9,10 $\neg$E
12	$\mid\ \mid\ \mid\ \mid\ {\sim} A_3$	6-11 $\sim$I
13	$\mid\ \mid\ \mid\ \neg A_2$	5-12 $\neg$I
14	$\mid\ \mid\ [(A \sqsupset \neg B) \sqsupset \neg A]_1$	3-13 $\sqsupset$I
15	$\mid\ ((A \sqsupset B) \sqsupset [(A \sqsupset \neg B) \sqsupset \neg A])_0$	1-14 $\sqsupset$I

A10 $\vdash_{NIL} \neg A \sqsupset (A \sqsupset B)$

1	$\vert\ \vert\ 0.1$	A (g, $\sqsupset$I)
2	$\vert\ \vert\ \underline{\neg A_1}$	
3	$\vert\ \vert\ \vert\ 1.2$	A (g, $\sqsupset$I)
4	$\vert\ \vert\ \vert\ \underline{A_2}$	
5	$\vert\ \vert\ \vert\ \vert\ \underline{\sim B_2}$	A (c, $\sim$E)
6	$\vert\ \vert\ \vert\ \vert\ A_2$	4 R
7	$\vert\ \vert\ \vert\ \vert\ \sim A_2$	2,3 $\neg$E
8	$\vert\ \vert\ \vert\ B_2$	5-7 $\sim$E
9	$\vert\ \vert\ (A \sqsupset B)_1$	3-8 $\sqsupset$I
10	$\vert\ [\neg A \sqsupset (A \sqsupset B)]_0$	1-9 $\sqsupset$I

A system with these axioms and MP (which we already have by AMρ with $\sqsupset$E) turns into classical logic if A10 is replaced by double negation, $\neg\neg A \sqsupset A$. But we cannot prove $\neg\neg A \sqsupset A$ (or at least we cannot if our derivation system is sound).

5.3 Soundness and Completeness

Preliminaries: Begin with generalized notions of validity to include expressions with subscripts and operator '$\sim$'. First, as a supplement to TIL,

TIL ($\sim$) $v_w(\sim A) = 1$ if $v_w(A) = 0$, and 0 otherwise.

For a model $\langle W, R, v\rangle$, let m be a map from subscripts into W. Say $\langle W, R, v\rangle_m$ is $\langle W, R, v\rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VIL* $\Gamma \models^*_{IL} A_s$ $[\![s.t]\!]$ iff there is no *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$.

NIL* $\Gamma \vdash^*_{NIL} A_s$ $[\![s.t]\!]$ iff there is an *NIL* derivation of A_s $[\![s.t]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so do not include expressions of the sort $s.t$) and do not include '$\sim$'. For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 5.1 *NIL is sound: If* $\Gamma \vdash_{NIL} A$ *then* $\Gamma \models_{IL} A$.

L5.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{IL} P_s$ $[\![s.t]\!]$, then $\Gamma' \models^*_{IL} P_s$ $[\![s.t]\!]$.

Reasoning parallel to that for L2.1 of $NK^{(t)}_{\alpha}$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NIL} \mathcal{P}$ then $\Gamma \models^*_{IL} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are formulas with subscript 0 and do not include '$\sim$'. Suppose $\Gamma \vdash^*_{NIL} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{IL} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $[\![s.t]\!]$. Then $\Gamma_1 = \{A_s\}$ $[\![\{s.t\}]\!]$; so for any $\langle W, R, v\rangle_m$, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $[\![\langle m(s), m(t)\rangle \in R]\!]$; so there is no $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $[\![\langle m(s), m(t)\rangle \notin R]\!]$. So by VIL*, $\Gamma_1 \models^*_{IL} A_s$ $[\![s.t]\!]$, where this is just to say, $\Gamma_1 \models^*_{IL} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^*_{IL} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\sim$I, $\sim$E, $\neg\!\!\!\rightarrow$I, $\neg\!\!\!\rightarrow$E, $\exists$I, $\exists$E, AMρ, AMτ or H. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{IL} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\wedge$I)

($\wedge$E)

($\vee$I)

($\vee$E)

($\sim$I) If $\mathcal{P}_k$ arises by $\sim$I, then the picture is like this,

$$\begin{array}{r|l|l} & & \underline{A_s} \\ & & \\ i & & B_t \\ j & & \sim B_t \\ k & \multicolumn{2}{l}{\sim A_s} \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\sim A_s$. By assumption, $\Gamma_i \models^*_{IL} B_t$ and $\Gamma_j \models^*_{IL} \sim B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_s\}$; so by L5.1, $\Gamma_k \cup \{A_s\} \models^*_{IL} B_t$ and $\Gamma_k \cup \{A_s\} \models^*_{IL} \sim B_t$. Suppose $\Gamma_k \not\models^*_{IL} \sim A_s$; then by VIL*, there is an *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\sim A) = 0$; so by TIL($\sim$), $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(s)}(A) = 1$; so $v_m(\Gamma_k \cup \{A_s\}) = 1$; so by VIL*, $v_{m(t)}(B) = 1$ and $v_{m(t)}(\sim B) = 1$; from the latter, by TIL($\sim$), $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} \sim A_s$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

($\sim$E)

($\neg$I) If $\mathcal{P}_k$ arises by $\neg$I, then the picture is like this,

$$\begin{array}{r|l|l} & & \underline{s.t} \\ & & \\ i & & \sim A_t \\ k & \multicolumn{2}{l}{\neg A_s} \end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{IL} \sim A_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t\}$; so by L5.1, $\Gamma_k \cup \{s.t\} \models^*_{IL} \sim A_t$. Suppose $\Gamma_k \not\models^*_{IL} \neg A_s$; then by VIL*, there is an *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TIL($\neg$), there is some $w \in W$ such that $m(s)Rw$ and $v_w(A) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VIL*, $v_{m'(t)}(\sim A) = 1$; so by TIL($\sim$), $v_{m'}(A) = 0$. But $m'(t) = w$; so $v_w(A) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} \neg A_s$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

($\neg$E) If $\mathcal{P}_k$ arises by $\neg$E, then the picture is like this,

$$\begin{array}{r|l} i & \neg A_s \\ j & s.t \\ & \\ k & {\sim}A_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is ${\sim}A_t$. By assumption, $\Gamma_i \models^*_{IL} \neg A_s$ and $\Gamma_j \models^*_{IL} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L5.1, $\Gamma_k \models^*_{IL} \neg A_s$ and $\Gamma_k \models^*_{IL} s.t$. Suppose $\Gamma_k \not\models^*_{IL} {\sim}A_t$; then by VIL*, there is some *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}({\sim}A) = 0$; so by TIL(${\sim}$), $v_{m(t)}(A) = 1$. Since $v_m(\Gamma_k) = 1$, by VIL*, $v_{m(s)}(\neg A) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, by TIL($\neg$), any w such that $m(s)Rw$ has $v_w(A) = 0$; so $v_{m(t)}(A) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} A_t$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

($\sqsupset$I) If $\mathcal{P}_k$ arises by $\sqsupset$I, then the picture is like this,

$$\begin{array}{r|l|l} & & s.t \\ & & \underline{A_t} \\ & & \\ i & & B_t \\ k & \multicolumn{2}{l}{(A \sqsupset B)_s} \end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $(A \sqsupset B)_s$. By assumption, $\Gamma_i \models^*_{IL} B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t, A_t\}$; so by L5.1, $\Gamma_k \cup \{s.t, A_t\} \models^*_{IL} B_t$. Suppose $\Gamma_k \not\models^*_{IL} (A \sqsupset B)_s$; then by VIL*, there is an *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(A \sqsupset B) = 0$; so by TIL($\sqsupset$), there is some $w \in W$ such that $m(s)Rw$ with $v_w(A) = 1$ and $v_w(B) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, v\rangle_{m'}$; since t does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; since $m'(t) = w$ and $m'(s) = m(s)$, $v_{m'(t)}(A) = 1$ and $\langle m'(s), m'(t)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t, A_t\}) = 1$; so by VIL*, $v_{m'(t)}(B) = 1$. But $m'(t) = w$; so $v_w(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} (A \sqsupset B)_s$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

($\sqsupset$E)

(AMρ)

(AMτ) If $\mathcal{P}_k$ arises by AMτ, then the picture is like this,

$$\begin{array}{r|l} i & s.t \\ j & t.u \\ & \\ k & s.u \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $s.u$. By assumption, $\Gamma_i \vDash^*_{IL} s.t$ and $\Gamma_j \vDash^*_{IL} t.u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L5.1, $\Gamma_k \vDash^*_{IL} s.t$ and $\Gamma_k \vDash^*_{IL} t.u$. Suppose $\Gamma_k \nvDash^*_{IL} s.u$; then by VIL*, there is some *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(u)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VIL*, $\langle m(s), m(t)\rangle \in R$ and $\langle m(t), m(u)\rangle \in R$; but *IL* includes condition τ; so for any $\langle x, y\rangle, \langle y, z\rangle \in R$, $\langle x, z\rangle \in R$; so $\langle m(s), m(u)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{IL} s.u$, which is to say, $\Gamma_k \vDash^*_{IL} \mathcal{P}_k$.

(H) If $\mathcal{P}_k$ arises by H, then the picture is like this,

$$\begin{array}{r|l} i & A_s \\ j & s.t \\ & \\ k & A_t \end{array}$$

where $i, j < k$, A has no instance of '$\sim$' and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \vDash^*_{IL} A_s$ and $\Gamma_j \vDash^*_{IL} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L5.1, $\Gamma_k \vDash^*_{IL} A_s$ and $\Gamma_k \vDash^*_{IL} s.t$. Suppose $\Gamma_k \nvDash^*_{IL} A_t$; then by VIL*, there is some *IL* interpretation $\langle W, R, v\rangle_m$ such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VIL*, $v_{m(s)}(A) = 1$ and $\langle m(s), m(t)\rangle \in R$.

Now, by induction on the number of operators in A, we show that for A without '$\sim$', if $v_x(A) = 1$ and xRy, then $v_y(A) = 1$. Suppose xRy.

Basis: Suppose A is a parameter p and $v_x(A) = 1$; then $v_x(p) = 1$; so by condition h, $v_y(p) = 1$; so $v_y(A) = 1$.

Assp: For $0 \leq i < k$, if A has i operators and $v_x(A) = 1$, then $v_y(A) = 1$.

Show: If A has k operators and $v_x(A) = 1$, then $v_y(A) = 1$.

If A has k operators and no instance of '$\sim$' then it is of the form, $P \wedge Q$, $P \vee Q$, $\neg P$, or $P \sqsupset Q$, where P and Q have $< k$ operators.

($\wedge$) Suppose A is $P \wedge Q$ and $v_x(A) = 1$; then $v_x(P \wedge Q) = 1$; so by TIL($\wedge$), $v_x(P) = 1$ and $v_x(Q) = 1$; so by assumption,

$v_y(P) = 1$ and $v_y(Q) = 1$; so by TIL($\wedge$), $v_y(P \wedge Q) = 1$; so $v_y(A) = 1$.

($\vee$) Suppose A is $P \vee Q$ and $v_x(A) = 1$; then $v_x(P \vee Q) = 1$; so by TIL($\vee$), $v_x(P) = 1$ or $v_x(Q) = 1$; so by assumption, $v_y(P) = 1$ or $v_y(Q) = 1$; so by TIL($\vee$), $v_y(P \vee Q) = 1$; so $v_y(A) = 1$.

($\neg$) Suppose A is $\neg P$ and $v_x(A) = 1$ but $v_y(A) = 0$; then $v_x(\neg P) = 1$ but $v_y(\neg P) = 0$. From the former, by TIL($\neg$), any w such that xRw has $v_w(P) = 0$. From the latter, by TIL($\neg$), there is some $z \in W$ such that yRz and $v_z(P) = 1$. But xRy and yRz so by τ, xRz; so $v_z(P) = 0$. This is impossible; reject the assumption: if $v_x(A) = 1$, then $v_y(A) = 1$.

($\sqsupset$) Suppose A is $P \sqsupset Q$ and $v_x(A) = 1$ but $v_y(A) = 0$; then $v_x(P \sqsupset Q) = 1$ but $v_y(P \sqsupset Q) = 0$. From the former, by TIL($\sqsupset$), any w such that xRw has $v_w(P) = 0$ or $v_w(Q) = 1$. From the latter, by TIL($\sqsupset$), there is some $z \in W$ such that yRz where $v_z(P) = 1$ and $v_z(Q) = 0$. But xRy and yRz so by τ, xRz; so $v_z(P) = 0$ or $v_z(Q) = 1$. This is impossible; reject the assumption: if $v_x(A) = 1$, then $v_y(A) = 1$.

For any such A, if $v_x(A) = 1$, then $v_y(A) = 1$.

So, returning to the case for (H), $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{IL} A_t$, which is to say, $\Gamma_k \models^*_{IL} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{IL} \mathcal{P}_i$.

Theorem 5.2 *NIL is complete: if* $\Gamma \models_{IL} A$ *then* $\Gamma \vdash_{NIL} A$.

Suppose $\Gamma \models_{IL} A$; then $\Gamma_0 \models^*_{IL} A_0$; we show that $\Gamma_0 \vdash^*_{NIL} A_0$. Again, this reduces to the standard notion.

Con Γ is consistent iff there is no A_s such that $\Gamma \vdash^*_{NIL} A_s$ and $\Gamma \vdash^*_{NIL} \sim A_s$.

L5.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NIL} \sim P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^*_{NIL} \sim P_s$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^*_{NIL} A_t$ and $\Gamma \cup \{P_s\} \vdash^*_{NIL} \sim A_t$. But then we can argue,

$$\begin{array}{r|l l}
1 & \Gamma & \\
2 & \quad \left| \underline{P_s} \right. & \text{A } (c, \sim\text{I}) \\
3 & \quad \left| A_t \right. & \text{from } \Gamma \cup \{P_s\} \\
4 & \quad \left| \sim A_t \right. & \text{from } \Gamma \cup \{P_s\} \\
5 & \sim P_s & \text{2-4 } \sim\text{I}
\end{array}$$

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^*_{NIL} \sim P_s$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NIL} \sim P_s$, then $\Gamma \cup \{P_s\}$ is consistent.

L5.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$

Proof by construction as for L2.3 of $NK\alpha$.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NIL} A_s$ or $\Gamma \vdash^*_{NIL} \sim A_s$.

SGT Γ is a SCAPEGOAT set for $\rightharpoondown$ iff for every formula of the form $\sim\rightharpoondown A_s$, if $\Gamma \vdash^*_{NIL} \sim\rightharpoondown A_s$ then there is some t such that $\Gamma \vdash^*_{NIL} s.t$ and $\Gamma \vdash^*_{NIL} A_t$.

Γ is a SCAPEGOAT set for $\sqsupset$ iff for every formula of the form $\sim(A \sqsupset B)_s$, if $\Gamma \vdash^*_{NIL} \sim(A \sqsupset B)_s$ then there is some t such that $\Gamma \vdash^*_{NIL} s.t$, $\Gamma \vdash^*_{NIL} A_t$ and $\Gamma \vdash^*_{NIL} \sim B_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L5.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first A_s in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without A_s, and set,

$$\begin{array}{ll}
\Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \vdash^*_{NIL} \sim A_s \\
\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\} & \text{if } \Omega_{i-1} \nvdash^*_{NIL} \sim A_s
\end{array}$$

and

$$\begin{array}{ll}
\Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } \sim\rightharpoondown P_s \text{ or } \sim(P \sqsupset Q)_s \\
\Omega_i = \Omega_{i^*} \cup \{s.t, P_t\} & \text{if } A_s \text{ is of the form } \sim\rightharpoondown P_s \\
\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \sim Q_t\} & \text{if } A_s \text{ is of the form } \sim(P \sqsupset Q)_s
\end{array}$$

-where t is the first subscript not included in Ω_{i^*}

then

$$\Gamma' = \bigcup_{i \geq 0} \Omega_i$$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be “considered” for inclusion at a subsequent stage.

L5.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NIL} A_s$ and $\Gamma' \nvdash^*_{NIL} {\sim}A_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NIL} {\sim}A_s$ then $\Gamma' \vdash^*_{NIL} {\sim}A_s$; but $\Gamma' \nvdash^*_{NIL} {\sim}A_s$; so $\Omega_{i-1} \nvdash^*_{NIL} {\sim}A_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NIL} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L5.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, or (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$, (iii) $\Omega_{k^*} \cup \{s.t, P_t\}$ or (iv) $\Omega_{k^*} \cup \{s.t, P_t, {\sim}Q_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{A_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NIL} {\sim}A_s$; so by L5.2, $\Omega_{k-1} \cup \{A_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, P_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, ${\sim}{\neg\!\!\!\!\rightarrow} P_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and ${\sim}A_u$ such that $\Omega_{k^*} \cup \{s.t, P_t\} \vdash^*_{NIL} A_u$ and $\Omega_{k^*} \cup \{s.t, P_t\} \vdash^*_{NIL} {\sim}A_u$. So reason as follows,

$$
\begin{array}{r|l|l}
1 & \Omega_{k^*} & \\
2 & \quad \underline{s.t} & \text{A } (g, {\neg\!\!\!\!\rightarrow}\text{I}) \\
3 & \quad\quad \underline{P_t} & \text{A } (c, {\sim}\text{I}) \\
4 & \quad\quad A_u & \text{from } \Omega_{k^*} \cup \{s.t, P_t\} \\
5 & \quad\quad {\sim}A_u & \text{from } \Omega_{k^*} \cup \{s.t, P_t\} \\
6 & \quad {\sim}P_t & \text{3-5 } {\sim}\text{I} \\
7 & {\neg\!\!\!\!\rightarrow} P_s & \text{2-6 } {\neg\!\!\!\!\rightarrow}\text{I}
\end{array}
$$

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NIL} \rightharpoondown P_s$; but $\sim\rightharpoondown P_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NIL} \sim\rightharpoondown P_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\sim(P \sqsupset Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_u and $\sim A_u$ such that $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\} \vdash^*_{NIL} A_u$ and $\Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\} \vdash^*_{NIL} \sim A_u$. So reason as follows,

$$\begin{array}{r|l l}
1 & \Omega_{k^*} & \\
2 & \mid\ s.t & \text{A } (g, \sqsupset\text{I}) \\
3 & \mid\ \underline{P_t} & \\
4 & \mid\ \mid\ \underline{\sim Q_t} & \text{A } (c, \sim\text{E}) \\
5 & \mid\ \mid\ A_u & \text{from } \Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\} \\
6 & \mid\ \mid\ \sim A_u & \text{from } \Omega_{k^*} \cup \{s.t, P_t, \sim Q_t\} \\
7 & \mid\ Q_t & \text{4-6 } \sim\text{E} \\
8 & (P \sqsupset Q)_s & \text{2-7 } \sqsupset\text{I}
\end{array}$$

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NIL} (P \sqsupset Q)_s$; but $\sim(P \sqsupset Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NIL} \sim(P \sqsupset Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L5.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 for $NK\alpha$.

L5.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $\rightharpoondown$ and $\sqsupset$.

For $\rightharpoondown$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NIL} \sim\rightharpoondown P_s$. By L5.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NIL} \sim\sim\rightharpoondown P_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\sim\rightharpoondown P_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, P_t\}$; so by construction, $s.t \in \Gamma'$ and $P_t \in \Gamma'$; so $\Gamma' \vdash^*_{NIL} s.t$ and $\Gamma' \vdash^*_{NIL} P_t$. So Γ' is a scapegoat set for $\rightharpoondown$.

For $\sqsupset$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NIL} \sim(P \sqsupset Q)_s$. By L5.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NIL} \sim\sim(P \sqsupset Q)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\sim(P \sqsupset Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \sim Q_t\}$; so by construction, $s.t \in \Gamma'$, $P_t \in \Gamma'$ and

$\sim Q_t \in \Gamma'$; so $\Gamma' \vdash^*_{NIL} s.t$, $\Gamma' \vdash^*_{NIL} P_t$ and $\Gamma' \vdash^*_{NIL} \sim Q_t$. So Γ' is a scapegoat set for $\sqsupset$.

C(I) We construct an interpretation $I = \langle W, R, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $\langle w_s, w_t\rangle \in R$ iff $\Gamma' \vdash^*_{NIL} s.t$, and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NIL} p_s$.

L5.8 If Γ_0 is consistent then for $\langle W, R, v\rangle$ constructed as above, and for any s included in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L5.4, Γ' is s-maximal. By L5.6 and L5.7, Γ' is consistent and a scapegoat set for $\rightharpoondown$ and $\sqsupset$. Now by induction on the number of operators in A_s,

Basis: If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NIL} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

Assp: For any i, $0 \leq i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

If A_s has k operators, then it is of the form $\sim P_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \sqsupset Q)_s$, or $\rightharpoondown P_s$ where P and Q have $< k$ operators.

($\sim$) A_s is $\sim P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\sim P) = 1$; so by TIL($\sim$), $v_{w_s}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NIL} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NIL} \sim P_s$, where this is to say, $\Gamma' \vdash^*_{NIL} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NIL} A_s$; then $\Gamma' \vdash^*_{NIL} \sim P_s$; so by consistency, $\Gamma' \nvdash^*_{NIL} P_s$; so by assumption, $v_{w_s}(P) = 0$; so by TIL($\sim$), $v_{w_s}(\sim P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NIL} A_s$.

($\wedge$)

($\vee$)

($\sqsupset$)

($\rightharpoondown$) A_s is $\rightharpoondown P_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NIL} A_s$; then $v_{w_s}(\rightharpoondown P) = 1$ but $\Gamma' \nvdash^*_{NIL} \rightharpoondown P_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NIL} \sim\rightharpoondown P_s$; so, since Γ' is a scapegoat set for $\rightharpoondown$, there is some t such that $\Gamma' \vdash^*_{NIL} s.t$ and $\Gamma' \vdash^*_{NIL} P_t$; from the first, by construction, $\langle w_s, w_t\rangle \in R$; and from the second, by assumption, $v_{w_t}(P) = 1$; so by TIL($\rightharpoondown$), $v_{w_s}(\rightharpoondown P) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$, then $\Gamma' \vdash^*_{NIL} A_s$.

(ii) Suppose $\Gamma' \vdash^{*}_{NIL} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^{*}_{NIL} \neg P_s$ but $v_{w_s}(\neg P) = 0$. From the latter, by TIL($\neg$), there is some $w_t \in W$ such that $w_s R w_t$ and $v_{w_t}(P) = 1$; so by assumption, $\Gamma' \vdash^{*}_{NIL} P_t$; but since $w_s R w_t$, by construction, $\Gamma' \vdash^{*}_{NIL} s.t$; so by ($\neg$E), $\Gamma' \vdash^{*}_{NIL} \sim P_t$; so by consistency, $\Gamma' \nvdash^{*}_{NIL} P_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NIL} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^{*}_{NIL} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^{*}_{NIL} A_s$.

L5.9 If Γ_0 is consistent, then $\langle W, R, v\rangle$ constructed as above is an *IL* interpretation.

For this, we need to show that the interpretation meets the ρ, τ and h conditions.

(ρ) Suppose $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^{*}_{NIL} s.s$; so by construction, $\langle w_s, w_s\rangle \in R$ and ρ is satisfied.

(τ)

(h) Suppose $v_{w_s}(p) = 1$ and $w_s R w_t$. Then by construction, $\Gamma' \vdash^{*}_{NIL} p_s$ and $\Gamma' \vdash^{*}_{NIL} s.t$; so by (H), $\Gamma' \vdash^{*}_{NIL} p_t$; so by construction, $v_{w_t}(p) = 1$.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L5.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 for *NK*α.

Main result: Suppose $\Gamma \models_{IL} A$ but $\Gamma \nvdash_{NIL} A$. Then $\Gamma_0 \models^{*}_{IL} A_0$ but $\Gamma_0 \nvdash^{*}_{NIL} A_0$. By a simple derivation, if $\Gamma_0 \vdash^{*}_{NIL} \sim\sim A_0$, then $\Gamma_0 \vdash^{*}_{NIL} A_0$; so $\Gamma_0 \nvdash^{*}_{NIL} \sim\sim A_0$; so by L5.2, $\Gamma_0 \cup \{\sim A_0\}$ is consistent; so by L5.9 and L5.10, there is an *IL* interpretation $\langle W, R, v\rangle_m$ constructed as above such that $v_m(\Gamma_0 \cup \{\sim A_0\}) = 1$; so $v_{m(0)}(\sim A) = 1$; so by TIL($\sim$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VIL*, $\Gamma_0 \not\models^{*}_{IL} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{IL} A$, then $\Gamma \vdash_{NIL} A$.

6 Many-Valued Logics: *Mx* (ch. 7,8)

6.1 Language / Semantic Notions

LMx The LANGUAGE consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, and $\supset$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \supset B)$. $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

IMx An INTERPRETATION is a function v which assigns to each propositional parameter some subset of $\{0, 1\}$; so $v(p)$ is ϕ, $\{1\}$, $\{0\}$ or $\{1, 0\}$. Intuitively, $v(p)$ is true iff $1 \in v(p)$ and $v(p)$ is false iff $0 \in v(p)$. Where x is empty or includes some combination of the following constraints,

exc	for no p are both $0 \in v(p)$ and $1 \in v(p)$	exclusion
exh	for any p, either $1 \in v(p)$ or $0 \in v(p)$	exhaustion

v is an *Mx* interpretation only if it meets the constraints from x. *Mcl* has both *exc* and *exh*, *Mk3* and *Mł3* just *exc*, *Mlp* and *Mrm* just *exh*, and *Mfd* neither *exc* nor *exh* (these are classical logic, and Priest's *K3*, *Ł3*, *LP*, *RM3* and *FDE*).

TM For complex expressions,

($\neg$) $1 \in v(\neg A)$ iff $0 \in v(A)$; $0 \in v(\neg A)$ iff $1 \in v(A)$.

($\wedge$) $1 \in v(A \wedge B)$ iff $1 \in v(A)$ and $1 \in v(B)$; $0 \in v(A \wedge B)$ iff $0 \in v(A)$ or $0 \in v(B)$.

($\vee$) $1 \in v(A \vee B)$ iff $1 \in v(A)$ or $1 \in v(B)$; $0 \in v(A \vee B)$ iff $0 \in v(A)$ and $0 \in v(B)$.

($\supset$) $1 \in v(A \supset B)$ iff $0 \in v(A)$ or $1 \in v(B)$; $0 \in v(A \supset B)$ iff $1 \in v(A)$ and $0 \in v(B)$.

$(\supset)_{L3}$ $1 \in v(A \supset B)$ iff $0 \in v(A)$ or $1 \in v(B)$ or none of $1, 0 \in v(A)$ or $1, 0 \in v(B)$; $0 \in v(A \supset B)$ iff $1 \in v(A)$ and $0 \in v(B)$.

$(\supset)_{RM}$ $1 \in v(A \supset B)$ iff $1 \notin v(A)$ or $0 \notin v(B)$ or all of $1, 0 \in v(A)$ and $1, 0 \in v(B)$; $0 \in v(A \supset B)$ iff $1 \in v(A)$ and $0 \in v(B)$.

All the systems have the same conditions, except that *Mł3* interpretations use $(\supset)_{L3}$ and *Mrm* interpretations use $(\supset)_{RM}$. For a set Γ of formulas, $1 \in v(\Gamma)$ iff $1 \in v(A)$ for each $A \in \Gamma$; then,

VMx $\Gamma \models_{Mx} A$ iff there is no *Mx* interpretation v such that $1 \in v(\Gamma)$ but $1 \notin v(A)$.

This account is adequate to the (superficially) different presentations in these chapters of Priest. For the multivalued approach: classical logic has values $\{0\}$, $\{1\}$, with $\{1\}$ designated; *K3* and *Ł3* have ϕ, $\{0\}$, $\{1\}$, with $\{1\}$ designated; *LP* and *RM3* have $\{0\}$, $\{1\}$, $\{0,1\}$, with $\{1\}$ and $\{0,1\}$ designated; and *FDE* has ϕ, $\{0\}$, $\{1\}$, $\{0,1\}$, with $\{1\}$ and $\{0,1\}$ designated. For the relational approach, we identify the relation as set membership. And a v as above maps to a Routley interpretation with $v_w(p) = 1$ iff $1 \in v(p)$, and $v_{w^*}(p) = 0$ iff $0 \in v(p)$.[5] Then, in each case, conditions for truth and validity are as above.

6.2 Natural Derivations: *NMx*

Introduce expressions of the sort A and $\overline{A}$. Intuitively $\overline{A}$ indicates that A is *not false*. Let $\backslash A\backslash$ and $/A/$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $\backslash A\backslash$ and $/A/$ are opposite. And similarly for $/\!\!/A/\!\!/$ and $\backslash\!\!\backslash A\backslash\!\!\backslash$, though there need be no fixed relation between overlines on $\backslash A\backslash$ and $\backslash\!\!\backslash A\backslash\!\!\backslash$. Except for a pair of new rules (D) and (U) corresponding to conditions *exc* and *exh*, derivation rules mirror ones for classical logic.

$$\begin{array}{r|l} \mathbf{D} & P \\ & \\ & \overline{P} \end{array} \qquad\qquad \begin{array}{r|l} \mathbf{U} & \overline{P} \\ & \\ & P \end{array}$$

$$\begin{array}{r|l} \mathbf{R} & /P/ \\ & \\ & /P/ \end{array} \qquad\qquad \begin{array}{r|l} \neg\mathbf{I} & \begin{array}{|l} \underline{/P/} \\ \\ /\!\!/Q/\!\!/ \\ \backslash\!\!\backslash\neg Q\backslash\!\!\backslash \end{array} \\ & \backslash\neg P\backslash \end{array} \qquad\qquad \begin{array}{r|l} \neg\mathbf{E} & \begin{array}{|l} \underline{/\neg P/} \\ \\ /\!\!/Q/\!\!/ \\ \backslash\!\!\backslash\neg Q\backslash\!\!\backslash \end{array} \\ & \backslash P\backslash \end{array}$$

$$\begin{array}{r|l} \wedge\mathbf{I} & /P/ \\ & /Q/ \\ & \\ & /P \wedge Q/ \end{array} \qquad\qquad \begin{array}{r|l} \wedge\mathbf{E} & /P \wedge Q/ \\ & \\ & /P/ \end{array} \qquad\qquad \begin{array}{r|l} \wedge\mathbf{E} & /P \wedge Q/ \\ & \\ & /Q/ \end{array}$$

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[5] For this, see [3, sections 8.5.8, 8.7.17 and 8.7.18] along with L6.0 for the proof of soundness in [7].

$$\begin{array}{r|l} \vee\textbf{I} & /P/ \\ & \\ & /P \vee Q/ \end{array} \qquad \begin{array}{r|l} \vee\textbf{I} & /P/ \\ & \\ & /Q \vee P/ \end{array} \qquad \begin{array}{r|l} \vee\textbf{E} & /P \vee Q/ \\ & \begin{array}{|l} \underline{/P/} \\ \\ /\!/R/\!/ \end{array} \\ & \begin{array}{|l} \underline{/Q/} \\ \\ /\!/R/\!/ \end{array} \\ & /\!/R/\!/ \end{array}$$

$$\begin{array}{r|l} \supset\textbf{I} & \begin{array}{|l} \underline{/P/} \\ \\ \backslash Q\backslash \end{array} \\ & \backslash P \supset Q\backslash \end{array} \qquad \begin{array}{r|l} \supset\textbf{E} & \backslash P \supset Q\backslash \\ & /P/ \\ & \\ & \backslash Q\backslash \end{array}$$

$$\begin{array}{r|l} \equiv\textbf{I} & \begin{array}{|l} \underline{/P/} \\ \\ \backslash Q\backslash \end{array} \\ & \begin{array}{|l} \underline{/Q/} \\ \\ \backslash P\backslash \end{array} \\ & \backslash P \equiv Q\backslash \end{array} \qquad \begin{array}{r|l} \equiv\textbf{E} & \backslash P \equiv Q\backslash \\ & /P/ \\ & \\ & \backslash Q\backslash \end{array} \qquad \begin{array}{r|l} \equiv\textbf{E} & \backslash P \equiv Q\backslash \\ & /Q/ \\ & \\ & \backslash P\backslash \end{array}$$

NMFD has the I- and E-rules for $\neg$, $\wedge$, $\vee$, $\supset$ with (R). *NMK3* adds (D), for truth *down*. *NMLP* adds (U), for truth *up*. *NMCL* has all the rules. In these systems, ($\equiv$I) and ($\equiv$E) are derived. In addition, for these systems, two-way derived rules carry over from *CL* with consistent overlines. Thus, e.g.,

$$\begin{array}{ll} \textbf{Impl} & /P \supset Q/ \;\lhd\rhd\; /\neg P \vee Q/ \\ & /\neg P \supset Q/ \;\lhd\rhd\; /P \vee Q/ \end{array}$$

MT, NB and DS appear in the forms,

$$\begin{array}{r|l} \textbf{MT} & /P \supset Q/ \\ & \backslash\neg Q\backslash \\ & \\ & /\neg P/ \end{array} \qquad \begin{array}{r|l} \textbf{NB} & /P \equiv Q/ \\ & \backslash\neg P\backslash \\ & \\ & /\neg Q/ \end{array} \quad \begin{array}{|l} /P \equiv Q/ \\ \backslash\neg Q\backslash \\ \\ /\neg P/ \end{array} \qquad \begin{array}{r|l} \textbf{DS} & /P \vee Q/ \\ & \backslash\neg P\backslash \\ & \\ & /Q/ \end{array} \quad \begin{array}{|l} /P \vee Q/ \\ \backslash\neg Q\backslash \\ \\ /P/ \end{array}$$

Alternate systems. The systems *NML3* and *NMRM* have (R) with I and E rules for $\neg$, $\wedge$, and $\vee$. Both include,

$$\begin{array}{r|l} \overline{\supset\text{I}} & \begin{array}{|l} \underline{P} \\ \\ \overline{Q} \end{array} \\ & \overline{P \supset Q} \end{array} \qquad \begin{array}{r|l} \overline{\supset\text{E}} & \overline{P \supset Q} \\ & P \\ & \\ & \overline{Q} \end{array}$$

which are the same as before. *NML3* adds (D) and,

$$\supset\mathrm{I}_{L3}\ \begin{array}{|l} \begin{array}{|l} \overline{P} \\ \underline{(P\vee\neg P)\vee(Q\vee\neg Q)} \\ \\ Q \end{array} \\ P\supset Q \end{array} \qquad\qquad \supset\mathrm{E}_{L3}\ \begin{array}{|l} P\supset Q \\ (P\vee\neg P)\vee(Q\vee\neg Q) \\ \overline{P} \\ \\ Q \end{array}$$

NMRM adds (U) and,

$$\supset\mathrm{I}_{RM}\ \begin{array}{|l} \begin{array}{|l} \underline{P \qquad\qquad\qquad\qquad} \\ \underline{(P\vee\neg P)\vee(Q\vee\neg Q)} \\ \\ \overline{Q} \end{array} \\ P\supset Q \end{array} \qquad\qquad \supset\mathrm{E}_{RM}\ \begin{array}{|l} \underline{P\supset Q \qquad\qquad\qquad} \\ \underline{(P\vee\neg P)\vee(Q\vee\neg Q)} \\ P \\ \\ \overline{Q} \end{array}$$

Because of the lack of symmetry for $\supset$ rules, there is no easy carryover in these systems of derived rules for $\equiv$ and $\supset$.

Where the members of Γ and A are expressions without overlines,

NMx $\Gamma \vdash_{NMx} A$ iff there is an *NMx* derivation of A from the members of Γ.

Examples. Here are derivations, cast to show the general forms, for MT and the second form of DS.

$$/P\supset Q/, \backslash\neg Q\backslash \vdash_{NMx} /\neg P/$$

$$\begin{array}{rll}
1 & \mid /P\supset Q/ & \text{P}\\
2 & \mid \underline{\backslash\neg Q\backslash} & \text{P}\\
3 & \mid\ \mid \underline{\backslash P\backslash} & \text{A } (c,\ \neg\text{I})\\
4 & \mid\ \mid /Q/ & 1{,}3 \supset\text{E}\\
5 & \mid\ \mid \backslash\neg Q\backslash & 2\ \text{R}\\
6 & \mid /\neg P/ & 3\text{-}5\ \neg\text{I}
\end{array}$$

$$/P\vee Q/, \backslash\neg Q\backslash \vdash_{NMx} /P/$$

$$\begin{array}{rll}
1 & \mid /P\vee Q/ & \text{P}\\
2 & \mid \underline{\backslash\neg Q\backslash} & \text{P}\\
3 & \mid\ \mid \underline{/P/} & \text{A } (g,\ 1\ \vee\text{E})\\
4 & \mid\ \mid /P/ & 3\ \text{R}\\
5 & \mid\ \mid \underline{/Q/} & \text{A } (g,\ 1\ \vee\text{E})\\
6 & \mid\ \mid\ \mid \underline{\backslash\neg P\backslash} & \text{A } (c,\ \neg\text{E})\\
7 & \mid\ \mid\ \mid /Q/ & 5\ \text{R}\\
8 & \mid\ \mid\ \mid \backslash\neg Q\backslash & 2\ \text{R}\\
9 & \mid\ \mid /P/ & 6\text{-}8\ \neg\text{E}\\
10 & \mid /P/ & 1{,}3\text{-}4{,}5\text{-}9\ \vee\text{E}
\end{array}$$

And for some particular results requiring (D) and (U), here are demonstrations of standard rule and axioms for classical logic, making use of the full rule set (see, e.g. [6, chapter 3]).

MP $A, A \supset B \vdash_{NM_{CL}} B$

$$\begin{array}{rll}
1 & \mid A & \text{P}\\
2 & \mid \underline{A \supset B} & \text{P}\\
3 & \mid \overline{A} & \text{1 D}\\
4 & \mid B & \text{2,3 } \supset\text{E}
\end{array}$$

A1 $\vdash_{NM_{CL}} A \supset (B \supset A)$

$$\begin{array}{rll}
1 & \mid \mid \underline{\overline{A}} & \text{A } (g, \supset\text{I})\\
2 & \mid \mid \mid \underline{\overline{B}} & \text{A } (g, \supset\text{I})\\
3 & \mid \mid \mid A & \text{1 U}\\
4 & \mid \mid B \supset A & \text{2-3 } \supset\text{I}\\
5 & \mid A \supset (B \supset A) & \text{1-4 } \supset\text{I}
\end{array}$$

A2 $\vdash_{NM_{CL}} [A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$

$$\begin{array}{rll}
1 & \mid \mid \underline{\overline{A \supset (B \supset C)}} & \text{A } (g, \supset\text{I})\\
2 & \mid \mid \mid \underline{\overline{A \supset B}} & \text{A } (g, \supset\text{I})\\
3 & \mid \mid \mid \mid \underline{\overline{A}} & \text{A } (g, \supset\text{I})\\
4 & \mid \mid \mid \mid A \supset B & \text{2 U}\\
5 & \mid \mid \mid \mid B & \text{3,4 } \supset\text{E}\\
6 & \mid \mid \mid \mid A \supset (B \supset C) & \text{1 U}\\
7 & \mid \mid \mid \mid B \supset C & \text{3,6 } \supset\text{E}\\
8 & \mid \mid \mid \mid \overline{B} & \text{5 D}\\
9 & \mid \mid \mid \mid C & \text{7,8 } \supset\text{E}\\
10 & \mid \mid \mid A \supset C & \text{3-9 } \supset\text{I}\\
11 & \mid \mid (A \supset B) \supset (A \supset C) & \text{2-10 } \supset\text{I}\\
12 & \mid [A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)] & \text{1-11 } \supset\text{I}
\end{array}$$

A3 $\vdash_{NM_{CL}} (\neg A \supset \neg B) \supset [(\neg A \supset B) \supset A]$

$$\begin{array}{rll}
1 & \mid \mid \underline{\overline{\neg A \supset \neg B}} & \text{A } (g, \supset\text{I})\\
2 & \mid \mid \mid \underline{\overline{\neg A \supset B}} & \text{A } (g, \supset\text{I})\\
3 & \mid \mid \mid \mid \underline{\overline{\neg A}} & \text{A } (c, \neg\text{E})\\
4 & \mid \mid \mid \mid \neg A & \text{3 U}\\
5 & \mid \mid \mid \mid \overline{B} & \text{2,4 } \supset\text{E}\\
6 & \mid \mid \mid \mid \overline{\neg B} & \text{1,4 } \supset\text{E}\\
7 & \mid \mid \mid \mid \neg B & \text{6 U}\\
8 & \mid \mid \mid A & \text{3-7 } \neg\text{E}\\
9 & \mid \mid (\neg A \supset B) \supset A & \text{2-8 } \supset\text{I}\\
10 & \mid (\neg A \supset \neg B) \supset [(\neg A \supset B) \supset A] & \text{1-9 } \supset\text{I}
\end{array}$$

Of course, there is not much point going back-and-forth between overline and non-overline expressions in the full classical system. But these examples should illustrate the rules. And overlines matter for the other systems. Finally, a couple derivations to show *modus ponens* as a derived rule in *NML3* and *NMRM*.

$P \supset Q, P \vdash_{NM_{L3}} Q$

1	$P \supset Q$	P
2	P	P
3	$\overline{P}$	2 D
4	$P \vee \neg P$	2 ∨I
5	$(P \vee \neg P) \vee (Q \vee \neg Q)$	4 ∨I
6	Q	1,3,5 ⊃E

$P \supset Q, P \vdash_{NM_{RM}} Q$

1	$P \supset Q$	P
2	P	P
3	$\overline{\neg Q}$	A (c, ¬E)
4	$\overline{Q \vee \neg Q}$	3 ∨I
5	$\overline{(P \vee \neg P) \vee (Q \vee \neg Q)}$	4 ∨I
6	$\overline{Q}$	1,2,5 ⊃E
7	$\neg Q$	3 U
8	Q	3-7 ¬E

6.3 Soundness and Completeness

Preliminaries: Begin with generalized notions of truth and validity to include expressions with overlines. First, *holding* as a generalization of TM. Say $/A/$ *holds* iff $h(A) = 1$ and otherwise *fails*. As usual, for the following, cases omitted are like ones worked, and so left to the reader.

HM (B) $h(p) = 1$ if $1 \in v(p)$, and otherwise $h(p) = 0$; $h(\overline{p}) = 1$ iff $0 \notin v(p)$, and otherwise $h(\overline{p}) = 0$.

($\neg$) $h(/\neg A/) = 1$ iff $h(\backslash A\backslash) = 0$, and otherwise $h(/\neg A/) = 0$.

($\wedge$) $h(/A \wedge B/) = 1$ iff $h(/A/) = 1$ and $h(/B/) = 1$, and otherwise $h(/A \wedge B/) = 0$.

($\vee$) $h(/A \vee B/) = 1$ iff $h(/A/) = 1$ or $h(/B/) = 1$, and otherwise $h(/A \vee B/) = 0$.

($\supset$) $h(/A \supset B/) = 1$ iff $h(\backslash A\backslash) = 0$ or $h(/B/) = 1$, and otherwise $h(/A \supset B/) = 0$.

$(\supset)_{L3}$ $h(A \supset B) = 1$ iff $h(\overline{A}) = 0$ or $h(B) = 1$ or none of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, or $h(\overline{B}) = 0$, and otherwise $h(A \supset B) =$

0; $h(\overline{A \supset B}) = 0$ iff $h(A) = 1$ and $h(\overline{B}) = 0$, and otherwise $h(\overline{A \supset B}) = 1$.

$(\supset)_{RM}$ $h(A \supset B) = 1$ iff $h(A) = 0$ or $h(\overline{B}) = 1$ or all of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, and $h(\overline{B}) = 0$, and otherwise $h(A \supset B) = 0$; $h(\overline{A \supset B}) = 0$ iff $h(A) = 1$ and $h(\overline{B}) = 0$, and otherwise $h(\overline{A \supset B}) = 1$.

Except for the $(\supset)_{L3}$ and $(\supset)_{RM}$ conditions, this formulation nicely mirrors the original classical definition TCL. And h and v are related as one would expect.

L6.0 For any Mx interpretation v and corresponding h, $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

Basis: If A has no operators, then it is a parameter p. By HM(B), $h(p) = 1$ iff $1 \in v(p)$ and $h(\overline{p}) = 1$ iff $0 \notin v(p)$; so $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

Assp: For $0 \leq i < k$, if A has k operators, then $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

Show: If A has k operators, then $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

If A has k operators, then it is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \supset Q$ where P and Q have $< k$ operators.

$(\neg)$ Suppose A is $\neg P$. By HM$(\neg)$, $h(\neg P) = 1$ iff $h(\overline{P}) = 0$; by assumption, iff $0 \in v(P)$; by TM$(\neg)$ iff $1 \in v(\neg P)$. By HM$(\neg)$, $h(\overline{\neg P}) = 1$ iff $h(P) = 0$; by assumption, iff $1 \notin v(P)$; by TM$(\neg)$ iff $0 \notin v(\neg P)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

$(\wedge)$ Suppose A is $P \wedge Q$. By HM$(\wedge)$, $h(P \wedge Q) = 1$ iff $h(P) = 1$ and $h(Q) = 1$; by assumption, iff $1 \in v(P)$ and $1 \in v(Q)$; by TM$(\wedge)$ iff $1 \in v(P \wedge Q)$. By HM$(\wedge)$, $h(\overline{P \wedge Q}) = 1$ iff $h(\overline{P}) = 1$ and $h(\overline{Q}) = 1$; by assumption, iff $0 \notin v(P)$ and $0 \notin v(Q)$; by TM$(\wedge)$ iff $0 \notin v(P \wedge Q)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

$(\vee)$

$(\supset)$ Suppose A is $P \supset Q$. By HM$(\supset)$, $h(P \supset Q) = 1$ iff $h(\overline{P}) = 0$ or $h(Q) = 1$; by assumption, iff $0 \in v(P)$ or $1 \in v(Q)$; by TM$(\supset)$ iff $1 \in v(P \supset Q)$. By HM$(\supset)$, $h(\overline{P \supset Q}) = 1$ iff

$h(P) = 0$ or $h(\overline{Q}) = 1$; by assumption, iff $1 \notin v(P)$ or $0 \notin v(Q)$; by TM($\supset$) iff $0 \notin v(P \supset Q)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

$(\supset)_{L3}$ Suppose A is $P \supset Q$. By HM$(\supset)_{L3}$, $h(P \supset Q) = 1$ iff $h(\overline{P}) = 0$ or $h(Q) = 1$ or none of $h(P) = 1$, $h(\overline{P}) = 0$, $h(Q) = 1$, or $h(\overline{Q}) = 0$; by assumption, iff $0 \in v(P)$ or $1 \in v(Q)$ or none of $1, 0 \in v(P)$ or $1, 0 \in v(Q)$; by TM$(\supset)_{L3}$ iff $1 \in v(P \supset Q)$. By HM$(\supset)_{L3}$, $h(\overline{P \supset Q}) = 1$ iff $h(P) = 0$ or $h(\overline{Q}) = 1$; by assumption, iff $1 \notin v(P)$ or $0 \notin v(Q)$; by TM$(\supset)_{L3}$ iff $0 \notin v(P \supset Q)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

$(\supset)_{RM}$ Suppose A is $P \supset Q$. By HM$(\supset)_{RM}$, $h(P \supset Q) = 1$ iff $h(P) = 0$ or $h(\overline{Q}) = 1$ or all of $h(P) = 1$, $h(\overline{P}) = 0$, $h(Q) = 1$, and $h(\overline{Q}) = 0$; by assumption, iff $1 \notin v(P)$ or $0 \notin v(Q)$ or all of $1, 0 \in v(P)$ and $1, 0 \in v(Q)$; by TM$(\supset)_{RM}$ iff $1 \in v(P \supset Q)$. By HM$(\supset)_{RM}$, $h(\overline{P \supset Q}) = 1$ iff $h(P) = 0$ or $h(\overline{Q}) = 1$; by assumption, iff $1 \notin v(P)$ or $0 \notin v(Q)$; by TM$(\supset)_{RM}$ iff $0 \notin v(P \supset Q)$. So $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

For any A, $h(A) = 1$ iff $1 \in v(A)$, and $h(\overline{A}) = 1$ iff $0 \notin v(A)$.

So A holds iff $1 \in v(A)$, and otherwise fails; and $\overline{A}$ holds iff $0 \notin v(A)$, and otherwise fails. This permits natural generalizations for notions of validity. For any v, where Γ is a set of expressions with or without overlines, say $h(\Gamma) = 1$ iff $h(/A/) = 1$ for each $/A/ \in \Gamma$. Then,

VMx* $\Gamma \models^{*}_{Mx} /A/$ iff there is no *Mx* interpretation v and corresponding h such that $h(\Gamma) = 1$ but $h(/A/) = 0$.

NMx* $\Gamma \vdash^{*}_{NMx} /A/$ iff there is an *NMx* derivation of $/A/$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and $/A/$ are without overlines. This is obvious for NMx*. And similarly, we have $h(A) = 1$ iff $1 \in v(A)$; so VMx* collapses to VMx.

THEOREM 6.1 *NMx is sound: If* $\Gamma \vdash_{NMx} A$ *then* $\Gamma \models_{Mx} A$.

L6.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^{*}_{Mx} /P/$, then $\Gamma' \models^{*}_{Mx} /P/$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Mx} /P/$, but $\Gamma' \not\models^*_{Mx} /P/$. From the latter, by VMx*, there is some v and h such that $h(\Gamma') = 1$ but $h(/P/) = 0$. But since $h(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $h(\Gamma) = 1$; so $h(\Gamma) = 1$ but $h(/P/) = 0$; so by VMx*, $\Gamma \not\models^*_{Mx} /P/$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Mx} /P/$, then $\Gamma' \models^*_{Mx} /P/$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the formula on line i (with or without overlines) and set Γ_i equal to the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NMx} /A/$ then $\Gamma \models^*_{Mx} /A/$. As above, this reduces to the standard result when the members of Γ and A are without overlines. Suppose $\Gamma \vdash^*_{NMx} /A/$. Then there is a derivation of $/A/$ from premises in Γ where $/A/$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{Mx} \mathcal{P}_i$. The case when $\mathcal{P}_i = /A/$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/$. Then $\Gamma_1 = \{/A/\}$; so $h(\Gamma_1) = 1$ iff $h(/A/) = 1$; so there is no h such that $h(\Gamma_1) = 1$ but $h(/A/) = 0$. So by VMx*, $\Gamma_1 \models^*_{Mx} /A/$, where this is just to say, $\Gamma_1 \models^*_{Mx} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^*_{Mx} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, ¬I, ¬E, ∧I, ∧E, ∨I, ∨E, or, depending on the system, ⊃I, ⊃E, D, U, $\overline{\supset\text{I}}$, $\overline{\supset\text{E}}$, $\supset\text{I}_{L3}$, $\supset\text{E}_{L3}$,$\supset\text{I}_{RM}$, or $\supset\text{E}_{RM}$. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

(¬I) If $\mathcal{P}_k$ arises by ¬I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & /A/ \\ \cline{3-3}
 & & \\
i & & /\!/B/\!/ \\
j & & \backslash\!\backslash\neg B\backslash\!\backslash \\
k & \multicolumn{2}{l}{\backslash\neg A\backslash}
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\backslash\neg A\backslash$. By assumption, $\Gamma_i \models^*_{Mx} /\!/B/\!/$ and $\Gamma_j \models^*_{Mx} \backslash\!\backslash\neg B\backslash\!\backslash$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{/A/\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{/A/\}$; so by L6.1, $\Gamma_k \cup \{/A/\} \models^*_{Mx} /\!/B/\!/$ and $\Gamma_k \cup \{/A/\} \models^*_{Mx}$

$\backslash\!\backslash\neg B\backslash\!\backslash$. Suppose $\Gamma_k \not\models^*_{Mx} \backslash\neg A\backslash$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(\backslash\neg A\backslash) = 0$; from the latter, by HM($\neg$), $h(/A/) = 1$; so $h(\Gamma_k) = 1$ and $h(/A/) = 1$; so $h(\Gamma_k \cup \{/A/\}) = 1$; so by VMx*, $h(/\!/B/\!/) = 1$ and $h(\backslash\!\backslash\neg B\backslash\!\backslash) = 1$; from the latter, by HM($\neg$), $h(/\!/B/\!/) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} \backslash\neg A\backslash$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

($\neg$E)

($\wedge$I)

($\wedge$E)

($\vee$I)

($\vee$E)

($\supset$I) If $\mathcal{P}_k$ arises by $\supset$I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & \backslash A\backslash \\ \cline{3-3}
 & & \\
i & & /B/ \\
k & \multicolumn{2}{l}{/A \supset B/}
\end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $/A \supset B/$. By assumption, $\Gamma_i \models^*_{Mx} /B/$; and by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{\backslash A\backslash\}$; so by L6.1, $\Gamma_k \cup \{\backslash A\backslash\} \models^*_{Mx} /B/$. Suppose $\Gamma_k \not\models^*_{Mx} /A \supset B/$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(/A \supset B/) = 0$; from the latter, by HM($\supset$), $h(\backslash A\backslash) = 1$ and $h(/B/) = 0$; so $h(\Gamma_k) = 1$ and $h(\backslash A\backslash) = 1$; so $h(\Gamma_k \cup \{\backslash A\backslash\}) = 1$; so by VMx*, $h(/B/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} /A \supset B/$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

($\supset$E) If $\mathcal{P}_k$ arises by $\supset$E, then the picture is like this,

$$\begin{array}{r|l}
i & /A \supset B/ \\
j & \backslash A\backslash \\
 & \\
k & /B/
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $/B/$. By assumption, $\Gamma_i \models^*_{Mx} /A \supset B/$ and $\Gamma_j \models^*_{Mx} \backslash A\backslash$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^*_{Mx} /A \supset B/$ and $\Gamma_k \models^*_{Mx} \backslash A\backslash$. Suppose $\Gamma_k \not\models^*_{Mx} /B/$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(/B/) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(/A \supset B/) = 1$ and $h(\backslash A\backslash) = 1$; from the former, by HM($\supset$), $h(\backslash A\backslash) = 0$ or $h(/B/) = 1$; so $h(/B/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} /B/$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

(D) If $\mathcal{P}_k$ arises by D, then the picture is like this,

$$\begin{array}{r|l} i & A \\ & \\ k & \overline{A} \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $\overline{A}$. Where this rule is included in *NMx*, *Mx* has condition *exc*, so no interpretation has $v(p) = \{1, 0\}$. By assumption, $\Gamma_i \models^*_{Mx} A$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^*_{Mx} A$. Suppose $\Gamma_k \not\models^*_{Mx} \overline{A}$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(\overline{A}) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(A) = 1$. But for these interpretations, for any A, if $h(A) = 1$ then $h(\overline{A}) = 1$.

Basis: A is a parameter p. Suppose $h(A) = 1$; then $h(p) = 1$; so by HM(B), $1 \in v(p)$; so by *exc*, $0 \notin v(p)$; so by HM(B), $h(\overline{p}) = 1$; so $h(\overline{A}) = 1$.

Assp: For any i, $0 \leq i < k$, if A has i operators, and $h(A) = 1$, then $h(\overline{A}) = 1$.

Show: If A has k operators, and $h(A) = 1$, then $h(\overline{A}) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \supset Q$, where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. Suppose $h(A) = 1$; then $h(\neg P) = 1$; so by HM($\neg$), $h(\overline{P}) = 0$; so by assumption, $h(P) = 0$; so by HM($\neg$), $h(\overline{\neg P}) = 1$, which is to say, $h(\overline{A}) = 1$.

($\wedge$) A is $P \wedge Q$. Suppose $h(A) = 1$; then $h(P \wedge Q) = 1$; so by HM($\wedge$), $h(P) = 1$ and $h(Q) = 1$; so by assumption, $h(\overline{P}) = 1$ and $h(\overline{Q}) = 1$; so by HM($\wedge$), $h(\overline{P \wedge Q}) = 1$, which is to say $h(\overline{A}) = 1$.

($\vee$)

($\supset$) A is $P \supset Q$. Suppose $h(A) = 1$; then $h(P \supset Q) = 1$; so by HM($\supset$), $h(\overline{P}) = 0$ or $h(Q) = 1$; so by assumption, $h(P) = 0$ or $h(\overline{Q}) = 1$; so by HM($\supset$), $h(\overline{P \supset Q}) = 1$, which is to say $h(\overline{A}) = 1$.

($\supset$)$_{L3}$ A is $P \supset Q$. Suppose $h(A) = 1$; then $h(P \supset Q) = 1$; so by HM($\supset$)$_{L3}$, $h(\overline{P}) = 0$ or $h(Q) = 1$ or none of $h(P) = 1$, $h(\overline{P}) = 0$, $h(Q) = 1$, or $h(\overline{Q}) = 0$; so by assumption, $h(P) = 0$

or $h(\overline{Q}) = 1$; so by HM($\supset$)$_{L3}$, $h(\overline{P \supset Q}) = 1$, which is to say $h(\overline{A}) = 1$.

For any A, if $h(A) = 1$, then $h(\overline{A}) = 1$.

So, returning to the case for (D), $h(\overline{A}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^{*}_{Mx} \overline{A}$, which is to say, $\Gamma_k \models^{*}_{Mx} \mathcal{P}_k$.

(U) If $\mathcal{P}_k$ arises by U, then the picture is like this,

$$\begin{array}{r|l} i & \overline{A} \\ & \\ k & A \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A. Where this rule is included in *NMx*, *Mx* has condition *exh*, so no interpretation has $v(p) = \phi$. By assumption, $\Gamma_i \models^{*}_{Mx} \overline{A}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^{*}_{Mx} \overline{A}$. Suppose $\Gamma_k \not\models^{*}_{Mx} A$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(A) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(\overline{A}) = 1$. But for these interpretations, for any A, if $h(\overline{A}) = 1$ then $h(A) = 1$.

Basis: A is a parameter p. Suppose $h(\overline{A}) = 1$; then $h(\overline{p}) = 1$; so by HM(B), $0 \notin v(p)$; so by *exh*, $1 \in v(p)$; so by HM(B), $h(p) = 1$; so $h(A) = 1$.

Assp: For any i, $0 \leq i < k$, if A has i operators, and $h(\overline{A}) = 1$, then $h(A) = 1$.

Show: If A has k operators, and $h(\overline{A}) = 1$, then $h(A) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \supset Q$, where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. Suppose $h(\overline{A}) = 1$; then $h(\overline{\neg P}) = 1$; so by HM($\neg$), $h(P) = 0$; so by assumption, $h(\overline{P}) = 0$; so by HM($\neg$), $h(\neg P) = 1$, which is to say, $h(A) = 1$.

($\wedge$) A is $P \wedge Q$. Suppose $h(\overline{A}) = 1$; then $h(\overline{P \wedge Q}) = 1$; so by HM($\wedge$), $h(\overline{P}) = 1$ and $h(\overline{Q}) = 1$; so by assumption, $h(P) = 1$ and $h(Q) = 1$; so by HM($\wedge$), $h(P \wedge Q) = 1$, which is to say $h(A) = 1$.

($\vee$)

($\supset$) A is $P \supset Q$. Suppose $h(\overline{A}) = 1$; then $h(\overline{P \supset Q}) = 1$; so by HM($\supset$), $h(P) = 0$ or $h(\overline{Q}) = 1$; so by assumption, $h(\overline{P}) = 0$ or $h(Q) = 1$; so by HM($\supset$), $h(P \supset Q) = 1$, which is to say $h(A) = 1$.

$(\supset)_{RM}$ A is $P \supset Q$. Suppose $h(\overline{A}) = 1$; then $h(\overline{P \supset Q}) = 1$; so by HM$(\supset)_{RM}$, $h(P) = 0$ or $h(\overline{Q}) = 1$; so $h(P) = 0$ or $h(\overline{Q}) = 1$ or all of $h(P) = 1$, $h(\overline{P}) = 0$, $h(Q) = 1$, and $h(\overline{Q}) = 0$; so by HM$(\supset)_{RM}$, $h(P \supset Q) = 1$, which is to say $h(A) = 1$.

For any A, if $h(\overline{A}) = 1$, then $h(A) = 1$.

So, returning to the case for (U), $h(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Mx} A$, which is to say, $\Gamma_k \models^*_{Mx} \mathcal{P}_k$.

$(\overline{\supset\mathrm{I}})$

$(\overline{\supset\mathrm{E}})$

$(\supset\mathrm{I}_{L3})$ If $\mathcal{P}_k$ arises by $\supset\mathrm{I}_{L3}$, then the picture is like this,

$$\begin{array}{r|l}
 & \begin{array}{|l} \overline{A} \\ \underline{(A \vee \neg A) \vee (B \vee \neg B)} \\ \\ \\ \end{array} \\
i & \begin{array}{|l} B \end{array} \\
k & A \supset B
\end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $A \supset B$. By assumption, $\Gamma_i \models^*_{NM_{L3}} B$; and by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{\overline{A}, (A \vee \neg A) \vee (B \vee \neg B)\}$; so by L6.1, $\Gamma_k \cup \{\overline{A}, (A \vee \neg A) \vee (B \vee \neg B)\} \models^*_{NM_{L3}} B$. Suppose $\Gamma_k \not\models^*_{NM_{L3}} A \supset B$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(A \supset B) = 0$; from the latter, by HM$(\supset)_{L3}$, $h(\overline{A}) = 1$ and $h(B) = 0$ and at least one of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, or $h(\overline{B}) = 0$; since at least one of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, or $h(\overline{B}) = 0$, by HM$(\neg)$ twice, at least one of $h(A) = 1$, $h(\neg A) = 1$, $h(B) = 1$, or $h(\neg B) = 1$; so by repeated applications of HM$(\vee)$, $h((A \vee \neg A) \vee (B \vee \neg B)) = 1$; so $h(\Gamma_k) = 1$, $h(\overline{A}) = 1$, and $h((A \vee \neg A) \vee (B \vee \neg B)) = 1$; so $h(\Gamma_k \cup \{\overline{A}, (A \vee \neg A) \vee (B \vee \neg B)\}) = 1$; so by VMx*, $h(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{NM_{L3}} A \supset B$, which is to say, $\Gamma_k \models^*_{NM_{L3}} \mathcal{P}_k$.

$(\supset\mathrm{E}_{L3})$ If $\mathcal{P}_k$ arises by $\supset\mathrm{E}_{L3}$, then the picture is like this,

$$\begin{array}{r|l}
h & A \supset B \\
i & (A \vee \neg A) \vee (B \vee \neg B) \\
j & \overline{A} \\
 & \\
k & B
\end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is B. By assumption, $\Gamma_h \models^*_{NM_{L3}} A \supset B$, $\Gamma_i \models^*_{NM_{L3}} (A \vee \neg A) \vee (B \vee \neg B)$ and $\Gamma_j \models^*_{NM_{L3}} \overline{A}$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^*_{NM_{L3}} A \supset B$, $\Gamma_k \models^*_{NM_{L3}} (A \vee \neg A) \vee (B \vee \neg B)$ and $\Gamma_k \models^*_{NM_{L3}} \overline{A}$. Suppose $\Gamma_k \not\models^*_{NM_{L3}} B$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(B) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(A \supset B) = 1$, $h((A \vee \neg A) \vee (B \vee \neg B)) = 1$ and $h(\overline{A}) = 1$; from the first of these, by HM$(\supset)_{L3}$, $h(\overline{A}) = 0$ or $h(B) = 1$ or none of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, or $h(\overline{B}) = 0$; but since $h((A \vee \neg A) \vee (B \vee \neg B)) = 1$, by repeated applications of HM$(\vee)$, at least one of $h(A) = 1$, $h(\neg A) = 1$, $h(B) = 1$, or $h(\neg B) = 1$; so by HM$(\neg)$ twice, at least one of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, or $h(\overline{B}) = 0$; so $h(\overline{A}) = 0$ or $h(B) = 1$; but since $h(\overline{A}) = 1$, $h(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{NM_{L3}} B$, which is to say, $\Gamma_k \models^*_{NM_{L3}} \mathcal{P}_k$.

($\supset$I$_{RM}$) If $\mathcal{P}_k$ arises by $\supset$I$_{RM}$, then the picture is like this,

$$
\begin{array}{r|l|l}
 & & A \\
 & & \overline{(A \vee \neg A) \vee (B \vee \neg B)} \\ \cline{3-3}
 & & \\
 & & \\
i & & \overline{B} \\
k & \multicolumn{2}{l}{A \supset B}
\end{array}
$$

where $i < k$ and $\mathcal{P}_k$ is $A \supset B$. By assumption, $\Gamma_i \models^*_{NM_{RM}} \overline{B}$; and by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A, \overline{(A \vee \neg A) \vee (B \vee \neg B)}\}$; so by L6.1, $\Gamma_k \cup \{A, \overline{(A \vee \neg A) \vee (B \vee \neg B)}\} \models^*_{NM_{RM}} \overline{B}$. Suppose $\Gamma_k \not\models^*_{NM_{RM}} A \supset B$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(A \supset B) = 0$; from the latter, by HM$(\supset)_{RM}$, $h(A) = 1$ and $h(\overline{B}) = 0$ and not all of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, and $h(\overline{B}) = 0$; since not all of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, or $h(\overline{B}) = 0$, at least one of $h(A) = 0$, $h(\overline{A}) = 1$, $h(B) = 0$, or $h(\overline{B}) = 1$; so by HM$(\neg)$ twice, at least one of $h(\overline{\neg A}) = 1$, $h(\overline{A}) = 1$, $h(\overline{\neg B}) = 1$, or $h(\overline{B}) = 1$; so by repeated applications of HM$(\vee)$, $h(\overline{(A \vee \neg A) \vee (B \vee \neg B)}) = 1$; so $h(\Gamma_k) = 1$, $h(A) = 1$, and $h(\overline{(A \vee \neg A) \vee (B \vee \neg B)}) = 1$; so $h(\Gamma_k \cup \{A, \overline{(A \vee \neg A) \vee (B \vee \neg B)}\}) = 1$; so by VMx*, $h(\overline{B}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{NM_{RM}} A \supset B$, which is to say, $\Gamma_k \models^*_{NM_{RM}} \mathcal{P}_k$.

($\supset$E$_{RM}$) If $\mathcal{P}_k$ arises by $\supset$E$_{RM}$, then the picture is like this,

$$\begin{array}{r|l} h & A \supset B \\ i & \overline{(A \vee \neg A) \vee (B \vee \neg B)} \\ j & A \\ & \\ k & \overline{B} \end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $\overline{B}$. By assumption, $\Gamma_h \models^{*}_{\mathit{NM}_{\mathit{RM}}} A \supset B$, $\Gamma_i \models^{*}_{\mathit{NM}_{\mathit{RM}}} \overline{(A \vee \neg A) \vee (B \vee \neg B)}$ and $\Gamma_j \models^{*}_{\mathit{NM}_{\mathit{RM}}} A$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L6.1, $\Gamma_k \models^{*}_{\mathit{NM}_{\mathit{RM}}} A \supset B$, $\Gamma_k \models^{*}_{\mathit{NM}_{\mathit{RM}}} \overline{(A \vee \neg A) \vee (B \vee \neg B)}$ and $\Gamma_k \models^{*}_{\mathit{NM}_{\mathit{RM}}} A$. Suppose $\Gamma_k \not\models^{*}_{\mathit{NM}_{\mathit{RM}}} \overline{B}$; then by VMx*, there is some v and h such that $h(\Gamma_k) = 1$ but $h(\overline{B}) = 0$; since $h(\Gamma_k) = 1$, by VMx*, $h(A \supset B) = 1$, $h(\overline{(A \vee \neg A) \vee (B \vee \neg B)}) = 1$ and $h(A) = 1$; from the first of these, by HM$(\supset)_{RM}$, $h(A) = 0$ or $h(\overline{B}) = 1$ or all of $h(A) = 1$, $h(\overline{A}) = 0$, $h(B) = 1$, and $h(\overline{B}) = 0$; but since $h(\overline{(A \vee \neg A) \vee (B \vee \neg B)}) = 1$, by repeated applications of HM$(\vee)$, at least one of $h(\overline{A}) = 1$, $h(\overline{\neg A}) = 1$, $h(\overline{B}) = 1$, or $h(\overline{\neg B}) = 1$; so by HM$(\neg)$ twice, at least one of $h(A) = 0$, $h(\overline{A}) = 1$, $h(B) = 0$, or $h(\overline{B}) = 1$; so $h(A) = 0$ or $h(\overline{B}) = 1$; but since $h(A) = 1$, $h(\overline{B}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^{*}_{\mathit{NM}_{\mathit{RM}}} \overline{B}$, which is to say, $\Gamma_k \models^{*}_{\mathit{NM}_{\mathit{RM}}} \mathcal{P}_k$.

For any i, $\Gamma_i \models^{*}_{\mathit{Mx}} A_i$.

THEOREM 6.2 *NMx is complete: if* $\Gamma \models_{\mathit{Mx}} A$ *then* $\Gamma \vdash_{\mathit{NMx}} A$.

Suppose $\Gamma \models_{\mathit{Mx}} A$; then $\Gamma \models^{*}_{\mathit{Mx}} A$; we show that $\Gamma \vdash^{*}_{\mathit{NMx}} A$. Again, this reduces to the standard notion when there are no overlines. Fix on some particular constraint(s) x. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A such that $\Gamma \vdash^{*}_{\mathit{NMx}} /A/$ and $\Gamma \vdash^{*}_{\mathit{NMx}} \backslash\neg A\backslash$.

L6.2 If $\Gamma \nvdash^{*}_{\mathit{NMx}} \backslash\neg P\backslash$, then $\Gamma \cup \{/P/\}$ is consistent.

Suppose $\Gamma \nvdash^{*}_{\mathit{NMx}} \backslash\neg P\backslash$ but $\Gamma \cup \{/P/\}$ is inconsistent. Then there is some A such that $\Gamma \cup \{/P/\} \vdash^{*}_{\mathit{NMx}} /\!/A/\!/$ and $\Gamma \cup \{/P/\} \vdash^{*}_{\mathit{NMx}} \backslash\!\backslash\neg A\backslash\!\backslash$. But then we can argue,

$$\begin{array}{r|l l}
1 & \Gamma & \\
2 & \quad \vert\ \underline{/P/} & \text{A } (c, \neg\text{I}) \\
3 & \quad \vert\ /\!/A/\!/ & \text{from } \Gamma \cup \{/P/\} \\
4 & \quad \vert\ \backslash\!\backslash\neg A\backslash\!\backslash & \text{from } \Gamma \cup \{/P/\} \\
5 & \backslash\neg P\backslash & \text{2-4 } \neg\text{I}
\end{array}$$

So $\Gamma \vdash^*_{NMx} \backslash\neg P\backslash$. But this is impossible; reject the assumption: if $\Gamma \nvdash^*_{NMx} \backslash\neg P\backslash$, then $\Gamma \cup \{/P/\}$ is consistent.

L6.3 There is an enumeration of all the formulas, $\mathcal{P}_1, \mathcal{P}_2 \ldots$

Proof by construction. A simple approach is to order A_1, $A_2 \ldots$ in the usual way, and let the final enumeration be, A_1, $\overline{A}_1$, A_2, $\overline{A}_2 \ldots$.

Max Γ is maximal iff for any A either $\Gamma \vdash^*_{NMx} /A/$ or $\Gamma \vdash^*_{NMx} \backslash\neg A\backslash$.

C(Γ') We construct a Γ' from Γ as follows. Set $\Omega_0 = \Gamma$. By L6.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the formulas; for any $\mathcal{P}_i = /A/$ in this series set,

$$\begin{array}{ll}
\Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \vdash^*_{NMx} \backslash\neg A\backslash \\
\Omega_i = \Omega_{i-1} \cup \{/A/\} & \text{if } \Omega_{i-1} \nvdash^*_{NMx} \backslash\neg A\backslash
\end{array}$$

then

$$\Gamma' = \bigcup_{i \geq 0} \Omega_i$$

L6.4 Γ' is maximal.

Suppose Γ' is not maximal. Then there is some $\mathcal{P}_i = /A/$ such that $\Gamma' \nvdash^*_{NMx} /A/$ and $\Gamma' \nvdash^*_{NMx} \backslash\neg A\backslash$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NMx} \backslash\neg A\backslash$ then $\Gamma' \vdash^*_{NMx} \backslash\neg A\backslash$; but $\Gamma' \nvdash^*_{NMx} \backslash\neg A\backslash$; so $\Omega_{i-1} \nvdash^*_{NMx} \backslash\neg A\backslash$; so by construction, $\Omega_i = \Omega_{i-1} \cup \{/A/\}$; so by construction, $/A/ \in \Gamma'$; so $\Gamma' \vdash^*_{NMx} /A/$. This is impossible; reject the assumption: Γ' is maximal.

L6.5 If Γ is consistent, then each Ω_i is consistent.

Suppose Γ is consistent.

Basis: $\Omega_0 = \Gamma$ and Γ is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either Ω_{k-1} or $\Omega_{k-1} \cup \{/A/\}$. Suppose the former; by assumption, Ω_{k-1} is consistent; so Ω_k is consistent. Suppose the latter; then by construction, $\Omega_{k-1} \nvdash^*_{NMx} \backslash\neg A\backslash$; so by L6.2, $\Omega_{k-1} \cup \{/A/\}$ is consistent; so Ω_k is consistent.

For any i, Ω_i is consistent.

L6.6 If Γ is consistent, then Γ' is consistent.

Suppose Γ is consistent, but Γ' is not; from the latter, there is some P such that $\Gamma' \vdash^*_{NMx} /P/$ and $\Gamma' \vdash^*_{NMx} \backslash\neg P\backslash$. Consider derivations D1 and D2 of these results and the premises of these derivations. Where $\mathcal{P}_i$ is the last of these premises in the enumeration of formulas, by the construction of Γ', each of the premises must be a member of Ω_i; so D1 and D2 are derivations from Ω_i; so Ω_i is not consistent. But since Γ is consistent, by L6.5, Ω_i is consistent. This is impossible; reject the assumption: if Γ is consistent then Γ' is consistent.

C(v) We construct an interpretation v based on Γ' as follows. For any parameter p, set $1 \in v(p)$ iff $\Gamma' \vdash^*_{NMx} p$, and $0 \in v(p)$ iff $\Gamma' \nvdash^*_{NMx} \overline{p}$.

L6.7 If Γ is consistent then for any A, $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

Suppose Γ is consistent. By L6.4, Γ' is maximal; by L6.6, Γ' is consistent. Now by induction on the number of operators in A,

Basis: If A has no operators, then it is a parameter p or $\overline{p}$. By construction, $\Gamma' \vdash^*_{NMx} p$ iff $1 \in v(p)$; by HM(B), iff $h(p) = 1$. Similarly, by construction, $\Gamma' \nvdash^*_{NMx} \overline{p}$ iff $0 \in v(p)$; by HM(B), iff $h(\overline{p}) \neq 1$. So $h(/p/) = 1$ iff $\Gamma' \vdash^*_{NMx} /p/$, which is to say, $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

Assp: For any i, $0 \leq i < k$, if A has i operators, then $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

Show: If A has k operators, then $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

If A has k operators, then it is of the form $\neg P$, $P \wedge Q$, $P \vee Q$ or $P \supset Q$ where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. (i) Suppose $h(/A/) = 1$; then $h(/\neg P/) = 1$; so by HM($\neg$), $h(\backslash P\backslash) = 0$; so by assumption, $\Gamma' \nvdash^*_{NMx} \backslash P\backslash$; so by maximality, $\Gamma' \vdash^*_{NMx} /\neg P/$, where this is to say, $\Gamma' \vdash^*_{NMx} /A/$. (ii) Suppose $\Gamma' \vdash^*_{NMx} /A/$; then $\Gamma' \vdash^*_{NMx} /\neg P/$; so by consistency, $\Gamma' \nvdash^*_{NMx} \backslash P\backslash$; so by assumption, $h(\backslash P\backslash) = 0$; so by HM($\neg$), $h(/\neg P/) = 1$, where this is to say, $h(/A/) = 1$. So $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

($\wedge$)

($\vee$)

($\supset$) A is $P \supset Q$. (i) Suppose $h(/A/) = 1$ but $\Gamma' \nvdash^*_{NMx} /A/$; then $h(/P \supset Q/) = 1$ but $\Gamma' \nvdash^*_{NMx} /P \supset Q/$. From the latter, by maximality, $\Gamma' \vdash^*_{NMx} \backslash\neg(P \supset Q)\backslash$; from this it follows, by the following derivations,

1	$\backslash\neg(P \supset Q)\backslash$	P
2	$\vert\ /\neg P/$	A (c, $\neg$E)
3	$\vert\ \vert\ \backslash P\backslash$	A (g, $\supset$I)
4	$\vert\ \vert\ \vert\ \backslash\neg Q\backslash$	A (c, $\neg$E)
5	$\vert\ \vert\ \vert\ \backslash P\backslash$	3 R
6	$\vert\ \vert\ \vert\ /\neg P/$	2 R
7	$\vert\ \vert\ /Q/$	4-6 $\neg$E
8	$\vert\ /P \supset Q/$	3-7 $\supset$I
9	$\vert\ \backslash\neg(P \supset Q)\backslash$	1 R
10	$\backslash P\backslash$	2-9 $\neg$E

1	$\backslash\neg(P \supset Q)\backslash$	P
2	$\vert\ /Q/$	A (c, $\neg$I)
3	$\vert\ \vert\ \backslash P\backslash$	A (g, $\supset$I)
4	$\vert\ \vert\ /Q/$	2 R
5	$\vert\ /P \supset Q/$	3-4 $\supset$I
6	$\vert\ \backslash\neg(P \supset Q)\backslash$	1 R
7	$\backslash\neg Q\backslash$	2-6 $\neg$I

that $\Gamma' \vdash^*_{NMx} \backslash P\backslash$ and $\Gamma' \vdash^*_{NMx} \backslash\neg Q\backslash$; so by consistency, $\Gamma' \nvdash^*_{NMx} /Q/$; so by assumption, $h(\backslash P\backslash) = 1$ and $h(/Q/) = 0$; so by HM($\supset$), $h(/P \supset Q/) = 0$. This is impossible; reject the assumption: if $h(/A/) = 1$ then $\Gamma' \vdash^*_{NMx} /A/$.

(ii) Suppose $\Gamma' \vdash^*_{NMx} /A/$ but $h(/A/) = 0$; then $\Gamma' \vdash^*_{NMx} /P \supset Q/$ but $h(/P \supset Q/) = 0$. From the latter, by HM($\supset$), $h(\backslash P\backslash) = 1$ and $h(/Q/) = 0$; so by assumption, $\Gamma' \vdash^*_{NMx} \backslash P\backslash$ and $\Gamma' \nvdash^*_{NMx} /Q/$; but since $\Gamma' \vdash^*_{NMx} /P \supset Q/$ and $\Gamma' \vdash^*_{NMx} \backslash P\backslash$, by ($\supset$E), $\Gamma' \vdash^*_{NMx} /Q/$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NMx} /A/$, then $h(/A/) = 1$. So $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

$(\supset)_{L3}$ A is $P \supset Q$. (i) $/A/$ is either (a) A or (b) $\overline{A}$. (a) Suppose $h(A) = 1$ but $\Gamma' \nvdash^*_{NM_{L3}} A$; then $h(P \supset Q) = 1$ but $\Gamma' \nvdash^*_{NM_{L3}} P \supset Q$. From the latter, by maximality, $\Gamma' \vdash^*_{NM_{L3}} \overline{\neg(P \supset Q)}$; from this it follows, by the following derivations,

$$\begin{array}{rll}
1 & \mid \overline{\neg(P \supset Q)} & \text{P} \\
2 & \mid \; \mid \neg P & \text{A } (c, \neg\text{E}) \\
3 & \mid \; \mid \; \mid \overline{P} & \text{A } (g, \supset\text{I}_{L3}) \\
4 & \mid \; \mid \; \mid (P \vee \neg P) \vee (Q \vee \neg Q) & \text{A } (g, \supset\text{I}_{L3}) \\
5 & \mid \; \mid \; \mid \; \mid \overline{\neg Q} & \text{A } (c, \neg\text{E}) \\
6 & \mid \; \mid \; \mid \; \mid \overline{P} & \text{3 R} \\
7 & \mid \; \mid \; \mid \; \mid \neg P & \text{2 R} \\
8 & \mid \; \mid \; \mid Q & \text{5-7 } \neg\text{E} \\
9 & \mid \; \mid P \supset Q & \text{3-8 } \supset\text{I}_{L3} \\
10 & \mid \; \mid \overline{\neg(P \supset Q)} & \text{1 R} \\
11 & \mid \overline{P} & \text{2-10 } \neg\text{E}
\end{array}$$

$$\begin{array}{rll}
1 & \mid \overline{\neg(P \supset Q)} & \text{P} \\
2 & \mid \; \mid Q & \text{A } (c, \neg\text{I}) \\
3 & \mid \; \mid \; \mid \overline{P} & \text{A } (g, \supset\text{I}_{L3}) \\
4 & \mid \; \mid \; \mid (P \vee \neg P) \vee (Q \vee \neg Q) & \text{A } (g, \supset\text{I}_{L3}) \\
5 & \mid \; \mid \; \mid Q & \text{2 R} \\
6 & \mid \; \mid P \supset Q & \text{3-5 } \supset\text{I}_{L3} \\
7 & \mid \; \mid \overline{\neg(P \supset Q)} & \text{1 R} \\
8 & \mid \overline{\neg Q} & \text{2-7 } \neg\text{I}
\end{array}$$

$$\begin{array}{rll}
1 & \mid \overline{\neg(P \supset Q)} & \text{P} \\
2 & \mid \; \mid \overline{\neg((P \vee \neg P) \vee (Q \vee \neg Q))} & \text{A } (c, \neg\text{E}) \\
3 & \mid \; \mid \; \mid \overline{P} & \text{A } (g, \supset\text{I}_{L3}) \\
4 & \mid \; \mid \; \mid (P \vee \neg P) \vee (Q \vee \neg Q) & \text{A } (g, \supset\text{I}_{L3}) \\
5 & \mid \; \mid \; \mid \; \mid \overline{\neg Q} & \text{A } (c, \neg\text{E}) \\
6 & \mid \; \mid \; \mid \; \mid (P \vee \neg P) \vee (Q \vee \neg Q) & \text{3 R} \\
7 & \mid \; \mid \; \mid \; \mid \overline{\neg((P \vee \neg P) \vee (Q \vee \neg Q))} & \text{2 R} \\
8 & \mid \; \mid \; \mid Q & \text{5-7 } \neg\text{E} \\
9 & \mid \; \mid P \supset Q & \text{3-8 } \supset\text{I}_{L3} \\
10 & \mid \; \mid \overline{\neg(P \supset Q)} & \text{1 R} \\
11 & \mid (P \vee \neg P) \vee (Q \vee \neg Q) & \text{2-10 } \neg\text{E}
\end{array}$$

that $\Gamma' \vdash^*_{NM_{L3}} \overline{P}$, $\Gamma' \vdash^*_{NM_{L3}} \overline{\neg Q}$, and $\Gamma' \vdash^*_{NM_{L3}} (P \vee \neg P) \vee (Q \vee \neg Q)$; from the second of these, by consistency, $\Gamma' \nvdash^*_{NM_{L3}} Q$; so by assumption, $h(\overline{P}) = 1$ and $h(Q) = 0$. Since $h(P \supset Q) = 1$, by HM($\supset$)$_{L3}$, $h(\overline{P}) = 0$ or $h(Q) = 1$ or none of $h(P) = 1$, $h(\overline{P}) = 0$, $h(Q) = 1$, or $h(\overline{Q}) = 0$; but since $h(\overline{P}) = 1$ and $h(Q) = 0$, none of $h(P) = 1$, $h(\overline{P}) = 0$, $h(Q) = 1$, or $h(\overline{Q}) = 0$;

so all of $h(P) = 0$, $h(\overline{P}) = 1$, $h(Q) = 0$, and $h(\overline{Q}) = 1$; so by assumption, $\Gamma' \nvdash^*_{NM_{L3}} P$, $\Gamma' \vdash^*_{NM_{L3}} \overline{P}$, $\Gamma' \nvdash^*_{NM_{L3}} Q$, and $\Gamma' \vdash^*_{NM_{L3}} \overline{Q}$; so by maximality, $\Gamma' \vdash^*_{NM_{L3}} \overline{P}$, $\Gamma' \vdash^*_{NM_{L3}} \overline{\neg P}$, $\Gamma' \vdash^*_{NM_{L3}} \overline{Q}$, and $\Gamma' \vdash^*_{NM_{L3}} \overline{\neg Q}$; from this, along with $\Gamma' \vdash^*_{NM_{L3}} (P \vee \neg P) \vee (Q \vee \neg Q)$, it follows, by the following derivation,

$$\begin{array}{rll}
1 & \mid \overline{P} & \text{P} \\
2 & \mid \overline{\neg P} & \text{P} \\
3 & \mid \overline{Q} & \text{P} \\
4 & \mid \overline{\neg Q} & \text{P} \\
5 & \mid (P \vee \neg P) \vee (Q \vee \neg Q) & \text{P} \\
6 & \mid \quad \mid \overline{\neg(P \supset Q)} & \text{A } (c, \neg\text{E}) \\
7 & \mid \quad \mid \overline{\neg\neg P} & 1 \text{ DN} \\
8 & \mid \quad \mid \overline{\neg\neg P} & 3 \text{ DN} \\
9 & \mid \quad \mid \overline{\neg P \wedge \neg\neg P} & 2,7 \wedge\text{I} \\
10 & \mid \quad \mid \overline{\neg Q \wedge \neg\neg Q} & 4,8 \wedge\text{I} \\
11 & \mid \quad \mid \neg(P \vee \neg P) & 9 \text{ DeM} \\
12 & \mid \quad \mid \neg(P \vee \neg P) & 10 \text{ DeM} \\
13 & \mid \quad \mid \neg(P \vee \neg P) \wedge \neg(Q \vee \neg Q) & 11,12 \wedge\text{I} \\
14 & \mid \quad \mid \overline{\neg((P \vee \neg P) \vee (Q \vee \neg Q))} & 13 \text{ DeM} \\
15 & \mid \quad \mid (P \vee \neg P) \vee (Q \vee \neg Q) & 5 \text{ R} \\
16 & \mid P \supset Q & \text{6-15 } \neg\text{E}
\end{array}$$

that $\Gamma' \vdash^*_{NM_{L3}} P \supset Q$. This is impossible; reject the assumption: if $h(A) = 1$ then $\Gamma' \vdash^*_{NM_{L3}} A$. (b) Suppose $h(\overline{A}) = 1$ but $\Gamma' \nvdash^*_{NM_{L3}} \overline{A}$; then $h(\overline{P \supset Q}) = 1$ but $\Gamma' \nvdash^*_{NM_{L3}} \overline{P \supset Q}$. From the former, by $\text{HM}(\supset)_{L3}$, $h(P) = 0$ or $h(\overline{Q}) = 1$; so by assumption, $\Gamma' \nvdash^*_{NM_{L3}} P$ or $\Gamma' \vdash^*_{NM_{L3}} \overline{Q}$. Suppose the first; then by maximality, $\Gamma' \vdash^*_{NM_{L3}} \overline{\neg P}$; from this it follows, by the following derivation,

$$\begin{array}{rll}
1 & \mid \overline{\neg P} & \text{P} \\
2 & \mid \quad \mid P & \text{A } (g, \overline{\supset}\text{I}) \\
3 & \mid \quad \mid \quad \mid \neg Q & \text{A } (c, \neg\text{E}) \\
4 & \mid \quad \mid \quad \mid P & 2 \text{ R} \\
5 & \mid \quad \mid \quad \mid \overline{\neg P} & 1 \text{ R} \\
6 & \mid \quad \mid \overline{Q} & \text{3-5 } \neg\text{E} \\
7 & \mid \overline{P \supset Q} & \text{2-6 } \overline{\supset}\text{I}
\end{array}$$

that $\Gamma' \vdash^*_{NM_{L3}} \overline{P \supset Q}$. This is impossible. Suppose the second; then $\Gamma' \vdash^*_{NM_{L3}} \overline{Q}$; from this it follows, by the following derivation,

$$
\begin{array}{r|lll}
1 & \overline{Q} & & \text{P} \\
\hline
2 & \quad | \; P & & \text{A } (g, \overline{\supset}\text{I}) \\
3 & \quad | \; \overline{Q} & & \text{1 R} \\
4 & \overline{P \supset Q} & & \text{2-3 } \overline{\supset}\text{I}
\end{array}
$$

that $\Gamma' \vdash^*_{NM_{L3}} \overline{P \supset Q}$. This is impossible; reject the assumption: if $h(\overline{A}) = 1$ then $\Gamma' \vdash^*_{NM_{L3}} \overline{A}$. If $h(/A/) = 1$ then $\Gamma' \vdash^*_{NM_{L3}} /A/$.

(ii) As before, $/A/$ is either (a) A or (b) $\overline{A}$. (a) Suppose $\Gamma' \vdash^*_{NM_{L3}} A$ but $h(A) = 0$; then $\Gamma' \vdash^*_{NM_{L3}} P \supset Q$ but $h(P \supset Q) = 0$. From the latter, by $\text{HM}(\supset)_{L3}$, $h(\overline{P}) = 1$ and $h(Q) = 0$ and at least one of $h(P) = 1$, $h(\overline{P}) = 0$, $h(Q) = 1$, or $h(\overline{Q}) = 0$; so $h(\overline{P}) = 1$, $h(Q) = 0$, and either $h(P) = 1$ or $h(\overline{Q}) = 0$; so by assumption, $\Gamma' \vdash^*_{NM_{L3}} \overline{P}$, $\Gamma' \nvdash^*_{NM_{L3}} \overline{Q}$, and either $\Gamma' \vdash^*_{NM_{L3}} P$ or $\Gamma' \nvdash^*_{NM_{L3}} \overline{Q}$. Suppose $\Gamma' \vdash^*_{NM_{L3}} P$; so $\Gamma' \vdash^*_{NM_{L3}} P \supset Q$, $\Gamma' \vdash^*_{NM_{L3}} P$, and $\Gamma' \vdash^*_{NM_{L3}} \overline{P}$; from this it follows, by the following derivation,

$$
\begin{array}{r|ll}
1 & P \supset Q & \text{P} \\
2 & P & \text{P} \\
3 & \overline{P} & \text{P} \\
\hline
4 & P \vee \neg P & \text{2 } \vee\text{I} \\
5 & (P \vee \neg P) \vee (Q \vee \neg Q) & \text{4 } \vee\text{I} \\
6 & Q & \text{1,5,3 } \supset\text{E}_{L3}
\end{array}
$$

that $\Gamma' \vdash^*_{NM_{L3}} Q$. This is impossible. Suppose $\Gamma' \nvdash^*_{NM_{L3}} \overline{Q}$; then by maximality, $\Gamma' \vdash^*_{NM_{L3}} \neg Q$; so $\Gamma' \vdash^*_{NM_{L3}} P \supset Q$, $\Gamma' \vdash^*_{NM_{L3}} \neg Q$, and $\Gamma' \vdash^*_{NM_{L3}} \overline{P}$; from this it follows, by the following derivation,

$$
\begin{array}{r|ll}
1 & P \supset Q & \text{P} \\
2 & \neg Q & \text{P} \\
3 & \overline{P} & \text{P} \\
\hline
4 & Q \vee \neg Q & \text{2 } \vee\text{I} \\
5 & (P \vee \neg P) \vee (Q \vee \neg Q) & \text{4 } \vee\text{I} \\
6 & Q & \text{1,5,3 } \supset\text{E}_{L3}
\end{array}
$$

that $\Gamma' \vdash^*_{NM_{L3}} Q$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NM_{L3}} A$ then $h(A) = 1$. (b) Suppose $\Gamma' \vdash^*_{NM_{L3}} \overline{A}$ but $h(\overline{A}) = 0$; then $\Gamma' \vdash^*_{NM_{L3}} \overline{P \supset Q}$ but $h(\overline{P \supset Q}) = 0$. From the latter, by $\text{HM}(\supset)_{L3}$, $h(P) = 1$ and $h(\overline{Q}) = 0$; so by assumption, $\Gamma' \vdash^*_{NM_{L3}} P$ and $\Gamma' \nvdash^*_{NM_{L3}} \overline{Q}$; so $\Gamma' \vdash^*_{NM_{L3}} \overline{P \supset Q}$ and $\Gamma' \vdash^*_{NM_{L3}} P$; from this it follows, by the following derivation,

$$\begin{array}{r|ll} 1 & \overline{P \supset Q} & \text{P} \\ 2 & \underline{P} & \text{P} \\ & & \\ 6 & \overline{Q} & \text{1,2 } \overline{\supset}\text{E} \end{array}$$

that $\Gamma' \vdash^*_{NM_{L3}} \overline{Q}$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NM_{L3}} \overline{A}$ then $h(\overline{A}) = 1$. If $\Gamma' \vdash^*_{NM_{L3}} /A/$ then $h(/A/) = 1$. So $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NM_{L3}} /A/$.

$(\supset)_{RM}$

For any A, $h(/A/) = 1$ iff $\Gamma' \vdash^*_{NMx} /A/$.

L6.8 If Γ is consistent, then v constructed as above is an *Mx* interpretation.

For this, we need to show that the relevant constraints are met. Suppose Γ is consistent; by L6.4, Γ' is maximal; by L6.6, Γ' is consistent.

(*exc*) For systems *MCL* and *MK3* with $v(p) \neq \{1, 0\}$, (D) is in *NKx*. Suppose $v(p) = \{1, 0\}$; then $1 \in v(p)$ and $0 \in v(p)$; so by construction, $\Gamma' \vdash^*_{NMx} p$ and $\Gamma' \nvdash^*_{NMx} \overline{p}$; from the latter, by maximality, $\Gamma' \vdash^*_{NMx} \neg p$; so by (D), $\Gamma' \vdash^*_{NMx} \overline{\neg p}$; so Γ' is inconsistent. This is impossible; reject the assumption: $v(p) \neq \{1, 0\}$.

(*exh*) For systems *MCL* and *MLP* with $v(p) \neq \phi$, (U) is in *NKx*. Suppose $v(p) = \phi$; then $1 \notin v(p)$ and $0 \notin v(p)$; so by construction, $\Gamma' \nvdash^*_{NMx} p$ and $\Gamma' \vdash^*_{NMx} \overline{p}$; from the former, by maximality, $\Gamma' \vdash^*_{NMx} \overline{\neg p}$; so by (U), $\Gamma' \vdash^*_{NMx} \neg p$; so Γ' is inconsistent. This is impossible; reject the assumption: $v(p) \neq \phi$.

L6.9 If Γ is consistent, then $h(\Gamma) = 1$.

Suppose Γ is consistent and $/A/ \in \Gamma$; then by construction, $/A/ \in \Gamma'$; so $\Gamma' \vdash^*_{NMx} /A/$; so since Γ is consistent, by L6.7, $h(/A/) = 1$. And similarly for any $/A/ \in \Gamma$. So $h(\Gamma) = 1$.

Main result: Suppose $\Gamma \models_{Mx} A$ but $\Gamma \nvdash_{NMx} A$. Then $\Gamma \models^*_{Mx} A$ but $\Gamma \nvdash^*_{NMx} A$. By (DN), if $\Gamma \vdash^*_{NMx} \neg\neg A$, then $\Gamma \vdash^*_{NMx} A$; so $\Gamma \nvdash^*_{NMx} \neg\neg A$; so by L6.2, $\Gamma \cup \{\overline{\neg A}\}$ is consistent; so by L6.8 and L6.9, there is an *Mx* interpretation v with corresponding h constructed as above such that $h(\Gamma \cup \{\overline{\neg A}\}) = 1$; so $h(\overline{\neg A}) = 1$; so by HM($\neg$), $h(A) = 0$; so $h(\Gamma) = 1$ and $h(A) = 0$; so by VMx*, $\Gamma \not\models^*_{Mx} A$. This is impossible; reject the assumption: if $\Gamma \models_{Mx} A$, then $\Gamma \vdash_{NMx} A$.

7 Gaps, Gluts and Worlds: $\upsilon\boldsymbol{X}$, ***Ix*** (ch. 9)

7.1 Language / Semantic Notions

This section is developed directly in terms introduced in demonstration of soundness and completeness in section 6. Apart from that discussion, the notions should be roughly familiar from derivations in that section.

LυX The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, and $\rightarrow$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \rightarrow B)$. $A \supset B$ abbreviates $\neg A \vee B$, and $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$. This time, from the start, for any formula A, we allow A and $\overline{A}$, where as before $/A/$ and $\backslash A\backslash$ ($/\!\!/A/\!\!/$ and $\backslash\!\!\backslash A\backslash\!\!\backslash$) represent one or the other (and similarly for N and $\overline{N}$ immediately below).

IυX An INTERPRETATION is $\langle W, N, \overline{N}, h\rangle$ where W is a set of worlds, and $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively; h is a function such that for any $w \in W$, $h_w(/p/) = 1$ or $h_w(/p/) = 0$, and for any w not in $/N/$, $h_w(/A \rightarrow B/) = 1$ or $h_w(/A \rightarrow B/) = 0$. So h makes assignments directly to expressions of the sort $/A \rightarrow B/$ at worlds not in $/N/$. Say $/A/$ *holds* at w if $h_w(/A/) = 1$ and otherwise *fails*. Interpretations may also be subject to the constraints,

K $\quad N = \overline{N} = W$
4 $\quad N = \overline{N}$

The K systems are subject to constraint (K), the 4 systems to (4). Of course, (K) implies (4); so it is enough that interpretations for υK_4 and υK_* are subject to (K); υN_4 is subject to (4), and υN_* to neither. With restriction K, h reduces to a simple assignment to parameters at worlds. Though it does not appear in Priest, we consider also a requirement (CL) which includes (4) and that for any for any $w \in N$, $h_w(p) = h_w(\overline{p})$.

Hυ For expressions not assigned a value directly,

($\neg$) $h_w(/\neg A/) = 1$ if $h_w(\backslash A\backslash) = 0$, and 0 otherwise.

($\wedge$) $h_w(/A \wedge B/) = 1$ if $h_w(/A/) = 1$ and $h_w(/B/) = 1$, and 0 otherwise.

($\vee$) $h_w(/A \vee B/) = 1$ if $h_w(/A/) = 1$ or $h_w(/B/) = 1$, and 0 otherwise.

$(\rightarrow)_4$ For $w \in /N/$, $h_w(/A \rightarrow B/) = 1$ iff there is no $x \in W$ such that $h_x(A) = 1$ and $h_x(/B/) = 0$.

$(\rightarrow)_*$ For $w \in /N/$, $h_w(/A \rightarrow B/) = 1$ iff there is no $x \in W$ such that $h_x(/\!\!/A/\!\!/) = 1$ and $h_x(/\!\!/B/\!\!/) = 0$.

The 4-systems υN_4 and υK_4 take H$\upsilon(\rightarrow)_4$; the star systems υN_* and υK_* take H$\upsilon(\rightarrow)_*$. Where Γ does not include formulas with overlines, $h_w(\Gamma) = 1$ iff $h_w(A) = 1$ for each $A \in \Gamma$; then,

VυX $\Gamma \models_{\upsilon X} A$ iff there is no υX interpretation $\langle W, N, \overline{N}, h\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ and $h_w(A) = 0$.

System Ix.

LIx The vocabulary is as before with $\sqsupset$ for $\rightarrow$. Again, for any formula A, allow A and $\overline{A}$.

IIx An interpretation is $\langle W, R, h\rangle$ where,

ρ	for all x, xRx	reflexivity
τ	for all x, y, z, if xRy and yRz then xRz	transitivity
h	for all x, y and p, if xRy, then if $h_x(p) = 1$, $h_y(p) = 1$, and if $h_y(\overline{p}) = 1$, $h_x(\overline{p}) = 1$	heredity

apply to any interpretation. In addition, interpretations may be subject to the condition,

exc	for no p are both $h(p) = 1$ and $h(\overline{p}) = 0$	exclusion

HIx is as before with,

$(\sqsupset)$ $h_x(A \sqsupset B) = 1$ iff there is no $y \in W$ such that xRy and $h_y(A) = 1$ but $h_y(B) = 0$. $h_x(\overline{A \sqsupset B}) = 1$ iff $h_x(A) = 0$ or $h_x(\overline{B}) = 1$.

$(\sqsupset)_W$ $h_x(A \sqsupset B) = 1$ iff there is no $y \in W$ such that xRy and $h_y(A) = 1$ but $h_y(B) = 0$. $h_x(\overline{A \sqsupset B}) = 1$ iff there is some $y \in W$ such that xRy and $h_y(A) = 1$ and $h_y(\overline{B}) = 0$.

The system I_4 takes neither *exc* nor $(\sqsupset)_W$. I_3 adds *exc*; IW adds to I_4 the $(\sqsupset)_W$ condition. Then validity works in the usual way.

As in the previous section, these accounts are meant to accommodate different presentations in Priest, and help exhibit their differences. In particular, as for the previous section, given constraint (4), an interpretation

$\langle W, N, \overline{N}, h\rangle$ corresponds to a relational $\langle W, N, \rho\rangle$, where $h_w(A) = 1$ iff A bears relation ρ (which, as in the previous section, may be set membership) to 1 at w, and $h_w(\overline{A}) = 1$ iff A does not bear ρ to 0 at w. And an interpretation $\langle W, N, \overline{N}, h\rangle$ corresponds to a star interpretation $\langle W, N, *, v\rangle$ where $h_w(A) = 1$ iff $v_w(A) = 1$ and $h_w(\overline{A}) = 1$ iff $v_{w^*}(A) = 1$.[6]

7.2 Natural Derivations: NvX, NIx

Allow expressions with both integer subscripts and overlines. $/n/[s]$ indicates that world s is an element of $/N/$. I- and E- rules for $\neg$, $\wedge$, $\vee$, $\supset$ and $\equiv$ are a natural combination of rules for *NKv* and *N*FDE, with rules for $\supset$ and $\equiv$ now derived.

$$\begin{array}{r|l} \mathbf{R} & /P/_s \\ & \\ & /P/_s \end{array}$$

$$\begin{array}{r|l} \neg\mathbf{I} & \begin{array}{|l} \underline{/P/_s} \\ \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash\neg P\backslash_s \end{array}$$

$$\begin{array}{r|l} \neg\mathbf{E} & \begin{array}{|l} \underline{/\neg P/_s} \\ \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash P\backslash_s \end{array}$$

$$\begin{array}{r|l} \wedge\mathbf{I} & /P/_s \\ & /Q/_s \\ & \\ & /P \wedge Q/_s \end{array}$$

$$\begin{array}{r|l} \wedge\mathbf{E} & /P \wedge Q/_s \\ & \\ & /P/_s \end{array}$$

$$\begin{array}{r|l} \wedge\mathbf{E} & /P \wedge Q/_s \\ & \\ & /Q/_s \end{array}$$

$$\begin{array}{r|l} \vee\mathbf{I} & /P/_s \\ & \\ & /P \vee Q/_s \end{array}$$

$$\begin{array}{r|l} \vee\mathbf{I} & /P/_s \\ & \\ & /Q \vee P/_s \end{array}$$

$$\begin{array}{r|l} \vee\mathbf{E} & /P \vee Q/_s \\ & \begin{array}{|l} \underline{/P/_s} \\ \\ /\!/R/\!/_t \end{array} \\ & \begin{array}{|l} \underline{/Q/_s} \\ \\ /\!/R/\!/_t \end{array} \\ & /\!/R/\!/_t \end{array}$$

$$\begin{array}{r|l} \supset\mathbf{I} & \begin{array}{|l} \underline{/P/_s} \\ \\ \backslash Q\backslash_s \end{array} \\ & \backslash P \supset Q\backslash_s \end{array}$$

$$\begin{array}{r|l} \supset\mathbf{E} & \backslash P \supset Q\backslash_s \\ & /P/_s \\ & \\ & \backslash Q\backslash_s \end{array}$$

---

[6] For the latter, given a star interpretation $\langle W, N, *, v\rangle$ consider an vX_* interpretation $\langle W', N', \overline{N}', h\rangle$ with a $w' \in W'$ corresponding to each $w \in W$. And for an vX_* interpretation $\langle W', N', \overline{N}', h\rangle$ consider a star interpretation $\langle W, N, *, v\rangle$ with a w and $w^* \in W$ corresponding to each $w' \in W'$. Then set $x' \in N'$ iff $x \in N$; $x' \in \overline{N}'$ iff $x^* \in N$; $h_{x'}(p) = 1$ iff $v_x(p) = 1$; $h_{x'}(\overline{p}) = 1$ iff $v_{x^*}(p) = 1$; for $x' \notin N'$, $h_{x'}(P \to Q) = 1$ iff $v_x(P \to Q) = 1$; and for $x' \notin \overline{N}'$, $h_{x'}(\overline{P \to Q}) = 1$ iff $v_{x^*}(P \to Q) = 1$. Then the result follows by a simple induction (for a related demonstration, see the proof of L7.0 in [7]).

$$\equiv\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} /P/_s \\ \hline \\ \backslash Q\backslash_s \end{array} \\ \begin{array}{|l} /Q/_s \\ \hline \\ \backslash P\backslash_s \end{array} \\ \backslash P \equiv Q\backslash_s \end{array} \qquad \equiv\mathbf{E}\ \begin{array}{|l} \backslash P \equiv Q\backslash_s \\ /P/_s \\ \\ \backslash Q\backslash_s \end{array} \qquad \equiv\mathbf{E}\ \begin{array}{|l} \backslash P \equiv Q\backslash_s \\ /Q/_s \\ \\ \backslash P\backslash_s \end{array}$$

The different derivation systems of this section add to these from,

$$\rightarrow\mathbf{I}_4\ \begin{array}{|l} /n/[s] \\ \begin{array}{|l} P_t \\ \hline \\ /Q/_t \end{array} \\ /P \rightarrow Q/_s \end{array} \qquad \rightarrow\mathbf{E}_4\ \begin{array}{|l} /n/[s] \\ /P \rightarrow Q/_s \\ P_t \\ \\ /Q/_t \end{array} \qquad \rightarrow\mathbf{I}{*}\ \begin{array}{|l} /n/[s] \\ \begin{array}{|l} /\!\!/P/\!\!/_t \\ \hline \\ /\!\!/Q/\!\!/_t \end{array} \\ /P \rightarrow Q/_s \end{array} \qquad \rightarrow\mathbf{E}{*}\ \begin{array}{|l} /n/[s] \\ /P \rightarrow Q/_s \\ /\!\!/P/\!\!/_t \\ \\ /\!\!/Q/\!\!/_t \end{array}$$

where t does not appear in any undischarged premise or assumption (→I4)

where t does not appear in any undischarged premise or assumption (→I*)

$$\mathbf{K}\ \begin{array}{|l} \\ \\ /n/[s] \end{array} \qquad \mathbf{NI}\ \begin{array}{|l} \\ \\ n[0] \end{array} \qquad \mathbf{Ca}\ \begin{array}{|l} /n/[s] \\ \\ \backslash n\backslash[s] \end{array} \qquad \mathbf{Cb}\ \begin{array}{|l} /n/[a] \\ /\!\!/P/\!\!/_a \\ \\ \backslash\!\!\backslash P\backslash\!\!\backslash_a \end{array}$$

For the star-rules, $/\!\!/P/\!\!/_t$ and $/\!\!/Q/\!\!/_t$ may be either P_t and Q_t, or $\overline{P}_t$ and $\overline{Q}_t$. Then,

NvK_4 adds →I4 and →E4 with K

NvK_* adds →I* and →E* with K

NvN_* adds →I* and →E* with NI

NvN_4 adds →I4 and →E4 with NI and Ca

A system with CL would add both Ca and Cb. As a simplification, in the first cases, one might eliminate rule K, and delete the normality requirement from other rules. In these systems, every subscript is 0, appears in a premise, or appears in the t-place of an accessible assumption for →I. Where the members of Γ and A are without overlines or subscripts, let Γ_0 be the members of Γ, each with subscript 0. Then,

NvX $\Gamma \vdash_{NvX} A$ iff there is an NvX derivation of A_0 from Γ_0.

Derived rules are as one would expect. Two-way derived rules carry over from *CL* with overlines and subscripts constant throughout. Thus, e.g.,

Impl $/P \supset Q/_s \;\lhd\rhd\; /\neg P \vee Q/_s$
$/\neg P \supset Q/_s \;\lhd\rhd\; /P \vee Q/_s$

MT, NB and DS appear in the forms,

$$\textbf{MT}\ \begin{array}{|l} /P \supset Q/_s \\ \backslash\neg Q\backslash_s \\ \\ /\neg P/_s \end{array} \qquad \textbf{NB}\ \begin{array}{|l} /P \equiv Q/_s \\ \backslash\neg P\backslash_s \\ \\ /\neg Q/_s \end{array} \quad \begin{array}{|l} /P \equiv Q/_s \\ \backslash\neg Q\backslash_s \\ \\ /\neg P/_s \end{array} \qquad \textbf{DS}\ \begin{array}{|l} /P \vee Q/_s \\ \backslash\neg P\backslash_s \\ \\ /Q/_s \end{array} \quad \begin{array}{|l} /P \vee Q/_s \\ \backslash\neg Q\backslash_s \\ \\ /P/_s \end{array}$$

System NIx. These systems take over rules for $\neg$, $\vee$ and $\wedge$ from before, and then add from the following in the natural way.

$$\sqsupset\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} s.t \\ \underline{P_t} \\ \\ Q_t \end{array} \\ (P \sqsupset Q)_s \end{array} \qquad\qquad \sqsupset\mathbf{E}\ \begin{array}{|l} (P \sqsupset Q)_s \\ s.t \\ P_t \\ \\ Q_t \end{array}$$

where t does not appear in any undischarged premise or assumption

$$\mathbf{AM}\rho\ \begin{array}{|l} \\ \\ s.s \end{array} \qquad \mathbf{AM}\tau\ \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array} \qquad \mathbf{H}_{\mathrm{I}}\ \begin{array}{|l} P_s \\ s.t \\ \\ P_t \end{array} \quad \begin{array}{|l} \overline{P}_t \\ s.t \\ \\ \overline{P}_s \end{array} \qquad \mathbf{D}\ \begin{array}{|l} P_s \\ \\ \overline{P}_s \end{array}$$

$$\overline{\sqsupset\mathrm{I}}\ \begin{array}{|l} \begin{array}{|l} \underline{P_s} \\ \\ \overline{Q}_s \end{array} \\ (\overline{P \sqsupset Q})_s \end{array} \qquad \overline{\sqsupset\mathrm{E}}\ \begin{array}{|l} (\overline{P \sqsupset Q})_s \\ P_s \\ \\ \overline{Q}_s \end{array} \qquad \overline{\sqsupset\mathrm{I}}_W\ \begin{array}{|l} s.t \\ P_t \\ \neg Q_t \\ \\ (\overline{P \sqsupset Q})_s \end{array} \qquad \overline{\sqsupset\mathrm{E}}_W\ \begin{array}{|l} (\overline{P \sqsupset Q})_s \\ \begin{array}{|l} s.t \\ P_t \\ \underline{\neg Q_t} \\ \\ A_u \end{array} \\ A_u \end{array}$$

where t does not appear in any undischarged premise or assumption and is not u

Each of the *NIx* systems have $\sqsupset$I, $\sqsupset$E, AMρ, AMτ and H$_I$. *NI*$_4$ then takes $\overline{\sqsupset\mathrm{I}}$ and $\overline{\sqsupset\mathrm{E}}$, *NI*$_3$ adds D. *NI*$_W$ substitutes $\overline{\sqsupset\mathrm{I}}_W$ and $\overline{\sqsupset\mathrm{E}}_W$ in the four-valued system. Validity is as before.

Examples. Here are a few cases where the logics do not all have the same results.

$P \to Q \vdash_{NvX_*} \neg Q \to \neg P$

1	$\vert\ (P \to Q)_0$	P
2	$\vert\ n[0]$	NI or K
3	$\vert\ \vert\ \neg Q_1$	A (g, →I∗)
4	$\vert\ \vert\ \vert\ \overline{P}_1$	A (c, ¬I)
5	$\vert\ \vert\ \vert\ \overline{Q}_1$	2,1,4 →E∗
6	$\vert\ \vert\ \vert\ \neg Q_1$	3 R
7	$\vert\ \vert\ \neg P_1$	4-6 ¬I
8	$\vert\ (\neg Q \to \neg P)_0$	2,3-7 →I∗

This derivation works with either (K) or (NI), but does not go through in the 4-systems insofar as there is no "purchase" for application of →E4 with (1) and only $\overline{P}_1$, rather than P_1, at (4).

$P \wedge \neg Q \vdash_{NvX_4} \neg(P \to Q)$

1	$\vert\ (P \wedge \neg Q)_0$	P
2	$\vert\ n[0]$	NI
3	$\vert\ \overline{n}[0]$	Ca or directly by K
4	$\vert\ \vert\ \overline{(P \to Q)}_0$	A (c, ¬I)
5	$\vert\ \vert\ P_0$	1 ∧E
6	$\vert\ \vert\ \overline{Q}_0$	3,4,5 →E4
7	$\vert\ \vert\ \neg Q_0$	1 ∧E
8	$\vert\ \neg(P \to Q)_0$	4-7 ¬I

This derivation works with either (NI) and (Ca) or (K). It is blocked in either star system insofar as the contradiction does not arise: by →E∗, we might get Q_0 at (4), but this does not contradict $\neg Q_0$ for ¬I.

$\vdash_{NvK_x} [(P \to Q) \wedge (Q \to R)] \to (P \to R)$

1	$\vert\ n[0]$	K
2	$\vert\ \vert\ [(P \to Q) \wedge (Q \to R)]_1$	A (g, →Ix)
3	$\vert\ \vert\ n[1]$	K
4	$\vert\ \vert\ \vert\ P_2$	A (g, →Ix)
5	$\vert\ \vert\ \vert\ (P \to Q)_1$	2 ∧E
6	$\vert\ \vert\ \vert\ Q_2$	3,4,5 →Ex
7	$\vert\ \vert\ \vert\ (Q \to R)_1$	2 ∧E
8	$\vert\ \vert\ \vert\ R_2$	3,6,7 →Ex
9	$\vert\ \vert\ (P \to R)_1$	3,4-8 →Ix
10	$\vert\ ([(P \to Q) \wedge (Q \to R)] \to (P \to R))_0$	1,2-9 →Ix

This derivation works with either the star- or 4-rules. But it works only with (K) insofar as $s = 1$ for lines (6), (8) and (9). And, finally, a couple cases to show $\neg(A \sqsupset B)_s \;\triangleleft\triangleright\; (A \sqsupset \neg B)_s$ in NI_W

$$\begin{array}{r|l l}
1 & \underline{\neg(A \sqsupset B)_s} & \text{P} \\
2 & \quad | \; s.t & \text{A } (g, \sqsupset\text{I}) \\
3 & \quad | \; \underline{A_t} & \\
4 & \quad | \quad | \; \underline{\overline{B}_t} & \text{A } (c, \neg\text{I}) \\
5 & \quad | \quad | \; \overline{(A \sqsupset B)}_s & \text{2,3,4 } \overline{\sqsupset\text{ I}} \\
6 & \quad | \quad | \; \neg(A \sqsupset B)_s & \text{1 R} \\
7 & \quad | \; \neg B_t & \text{4-6 } \neg\text{I} \\
8 & (A \sqsupset \neg B)_s & \text{2-7 } \sqsupset\text{I}
\end{array}$$

$$\begin{array}{r|l l}
1 & \underline{(A \sqsupset \neg B)_s} & \text{P} \\
2 & \quad | \; \underline{(\overline{A \sqsupset B})_s} & \text{A } (c, \neg\text{I}) \\
3 & \quad | \quad | \; s.t & \text{A } g, \text{ 2 } \overline{\sqsupset\text{ E}}) \\
4 & \quad | \quad | \; A_t & \\
5 & \quad | \quad | \; \underline{\overline{B}_t} & \\
6 & \quad | \quad | \quad | \; \underline{(\overline{A \sqsupset B})_s} & \text{A } \neg\text{I} \\
7 & \quad | \quad | \quad | \; \neg B_t & \text{1,3,4 } \sqsupset\text{E} \\
8 & \quad | \quad | \quad | \; \overline{B}_t & \text{5 R} \\
9 & \quad | \quad | \; \neg(A \sqsupset B)_s & \text{6-8 } \neg\text{I} \\
10 & \quad | \; \neg(A \sqsupset B)_s & \text{2,3-9 } \overline{\sqsupset\text{ E}} \\
11 & \quad | \; (\overline{A \sqsupset B})_s & \text{2 R} \\
12 & \neg(A \sqsupset B)_s & \text{2-11 } \neg\text{I}
\end{array}$$

7.3 Soundness and Completeness: υX

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, N, \overline{N}, h \rangle$, let m be a map from subscripts into W such that $m(0)$ is some member of N. Then say $\langle W, N, \overline{N}, h \rangle_m$ is $\langle W, N, \overline{N}, h \rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $h_m(\Gamma) = 1$ iff for each $/A_s/ \in \Gamma$, $h_{m(s)}(/A/) = 1$, and for each $/n/[s] \in \Gamma$, $m(s) \in /N/$. Now expand notions of validity for subscripts, overlines, and alternate expressions as indicated in double brackets as follows,

VυX* $\Gamma \models^*_{\upsilon X} /A/_s$ $[\![/n/[s]]\!]$ iff there is no υX interpretation $\langle W, N, \overline{N}, h \rangle_m$ such that $h_m(\Gamma) = 1$ but $h_{m(s)}(/A/) = 0$ $[\![m(s) \notin /N/]\!]$.

NυX* $\Gamma \vdash^*_{N\upsilon X} /A/_s$ $[\![/n/[s]]\!]$ iff there is an $N\upsilon X$ derivation of $/A/_s$ $[\![/n/[s]]\!]$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A are without overlines and have subscript 0 (and so do not include expressions of the sort $/n/[s]$). As usual, for the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 7.1 *$N\upsilon X$ is sound: If $\Gamma \vdash_{N\upsilon X} A$ then $\Gamma \models_{\upsilon X} A$.*

For the $(\rightarrow)_*$ case, it will be useful to have a further preliminary.

L7.0 For an interpretation $\langle W, N, \overline{N}, h\rangle$, consider $\langle W', N', \overline{N}', h'\rangle$ such that corresponding to each $w \in W$ there are $w', w^* \in W'$ where, (i) $w' \in /N'/$ iff $w \in /N/$, and $w* \in /N'/$ iff $w \in \backslash N\backslash$; (ii) $h'_{w'}(/p/) = 1$ iff $h_w(/p/) = 1$, and $h'_{w^*}(/p/) = 1$ iff $h_w(\backslash p\backslash) = 1$; (iii) for $w' \notin /N'/$, $h'_{w'}(/P \rightarrow Q/) = 1$ iff $h_w(/P \rightarrow Q/) = 1$, and for $w^* \notin /N'/$, $h'_{w^*}(/P \rightarrow Q/) = 1$ iff $h_w(\backslash P \rightarrow Q\backslash) = 1$. Then,

For the star systems and interpretations as above, for any $/A/$, (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

Basis: $/A/$ is an atomic $/p/$. (i) By construction, $h'_{w'}(/p/) = 1$ iff $h_w(/p/) = 1$; so $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$. Similarly, (ii) by construction, $h'_{w^*}(/p/) = 1$ iff $h_w(\backslash p\backslash) = 1$; so $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

Assp: For any i, $0 \leq i < k$, if $/A/$ has i operators, (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

Show: If $/A/$ has k operators, then (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

If $/A/$ has k operators, then it is of the form, $/\neg P/$, $/P \wedge Q/$, $/P \vee Q/$, or $/P \rightarrow Q/$, where P and Q have $< k$ operators.

($\neg$) $/A/$ is $/\neg P/$. (i) $h'_{w'}(/A/) = 1$ iff $h'_{w'}(/\neg P/) = 1$; by H$\upsilon(\neg)$, iff $h'_{w'}(\backslash P\backslash) = 0$; by assumption iff $h_w(\backslash P\backslash) = 0$; by H$\upsilon(\neg)$, iff $h_w(/\neg P/) = 1$; iff $h_w(/A/) = 1$. (ii) $h'_{w^*}(/A/) = 1$ iff $h'_{w^*}(/\neg P/) = 1$; by H$\upsilon(\neg)$, iff $h'_{w^*}(\backslash P\backslash) = 0$; by assumption iff $h_w(/P/) = 0$; by H$\upsilon(\neg)$, iff $h_w(\backslash\neg P\backslash) = 1$; iff $h_w(\backslash A\backslash) = 1$.

($\wedge$) $/A/$ is $/P \wedge Q/$. (i) $h'_{w'}(/A/) = 1$ iff $h'_{w'}(/P \wedge Q/) = 1$; by H$\upsilon(\wedge)$, iff $h'_{w'}(/P/) = 1$ and $h'_{w'}(/Q/) = 1$; by assumption, iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$; by H$\upsilon(\wedge)$, iff $h_w(/P \wedge Q/) = 1$; iff $h_w(/A/) = 1$. (ii) $h'_{w^*}(/A/) = 1$ iff $h'_{w^*}(/P \wedge Q/) = 1$; by H$\upsilon(\wedge)$, iff $h'_{w^*}(/P/) = 1$ and $h'_{w^*}(/Q/) = 1$; by assumption, iff $h_w(\backslash P\backslash) = 1$ and $h_w(\backslash Q\backslash) = 1$; by H$\upsilon(\wedge)$, iff $h_w(\backslash P \wedge Q\backslash) = 1$; iff $h_w(\backslash A\backslash) = 1$.

($\vee$)

($\rightarrow$) A is $/P \rightarrow Q/$. (i) Suppose $w' \notin /N'/$; then by construction, $h'_{w'}(/P \rightarrow Q/) = 1$ iff $h_w(/P \rightarrow Q/) = 1$; so $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$. So suppose $w' \in /N'/$; then by construction, $w \in /N/$. $h'_{w'}(/A/) = 0$ iff $h'_{w'}(/A \rightarrow B/) = 0$; since $w' \in /N'/$, by $\mathrm{H}\upsilon(\rightarrow)_*$ iff either there is an $x' \in W'$ such that $h'_{x'}(\sslash P\sslash) = 1$ and $h'_{x'}(\sslash Q\sslash) = 0$, or there is a $y^* \in W'$ such that $h'_{y^*}(\sslash P\sslash) = 1$ and $h'_{y^*}(\sslash Q\sslash) = 0$; by assumption, iff either $h_x(\sslash P\sslash) = 1$ and $h_x(\sslash Q\sslash) = 0$, or $h_y(\bbslash P\bbslash) = 1$ and $h_y(\bbslash Q\bbslash) = 0$; given either of these, since $w \in /N/$, by $\mathrm{H}\upsilon(\rightarrow)_*$, iff $h_w(/P \rightarrow Q/) = 0$; iff $h_w(/A/) = 0$.

(ii) Suppose $w^* \notin /N'/$; then by construction, $h'_{w^*}(/P \rightarrow Q/) = 1$ iff $h_w(\backslash P \rightarrow Q\backslash) = 1$; so $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$. So suppose $w^* \in /N'/$; then $w \in \backslash N\backslash$. $h'_{w^*}(/A/) = 0$ iff $h'_{w^*}(/A \rightarrow B/) = 0$; since $w^* \in /N'/$, by $\mathrm{H}\upsilon(\rightarrow)_*$ iff either there is an $x' \in W'$ such that $h'_{x'}(\sslash P\sslash) = 1$ and $h'_{x'}(\sslash Q\sslash) = 0$, or there is a $y^* \in W'$ such that $h'_{y^*}(\sslash P\sslash) = 1$ and $h'_{y^*}(\sslash Q\sslash) = 0$; by assumption, iff either $h_x(\sslash P\sslash) = 1$ and $h_x(\sslash Q\sslash) = 0$, or $h_y(\bbslash P\bbslash) = 1$ and $h_y(\bbslash Q\bbslash) = 0$; given either of these, since $w \in \backslash N\backslash$, by $\mathrm{H}\upsilon(\rightarrow)_*$, iff $h_w(\backslash P \rightarrow Q\backslash) = 0$; iff $h_w(\backslash A\backslash) = 0$.

For any A, (i) $h'_{w'}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $h'_{w^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

L7.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{\upsilon X} /P/_s \llbracket/n/[s]\rrbracket$ then $\Gamma' \models^*_{\upsilon X} /P/_s \llbracket/n/[s]\rrbracket$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{\upsilon X} /P/_s \llbracket/n/[s]\rrbracket$, but $\Gamma' \not\models^*_{\upsilon X} /P/_s \llbracket/n/[s]\rrbracket$. From the latter, by $\mathrm{V}\upsilon\mathrm{X}^*$, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma') = 1$ but $h_{m(s)}(/P/) = 0$ $\llbracket m(s) \notin /N/\rrbracket$. But since $h_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $h_m(\Gamma) = 1$; so $h_m(\Gamma) = 1$ but $h_{m(s)}(/P/) = 0$ $\llbracket m(s) \notin /N/\rrbracket$; so by $\mathrm{V}\upsilon\mathrm{X}^*$, $\Gamma \not\models^*_{\upsilon X} /P/_s \llbracket/n/[s]\rrbracket$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{\upsilon X} /P/_s \llbracket/n/[s]\rrbracket$, then $\Gamma' \models^*_{\upsilon X} /P/_s \llbracket/n/[s]\rrbracket$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{N\upsilon X} \mathcal{P}$ then $\Gamma \models^*_{\upsilon X} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are without overlines and have subscript 0. Suppose $\Gamma \vdash^*_{N\upsilon X} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{\upsilon X} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/_s$ $\llbracket /n/[s] \rrbracket$. Then $\Gamma_1 = \{/A/_s\}$ $\llbracket \{/n/[s]\} \rrbracket$; so for any $\langle W, N, \overline{N}, h \rangle_m$, $h_m(\Gamma_1) = 1$ iff $h_{m(s)}(/A/) = 1$ $\llbracket m(s) \in /N/ \rrbracket$; so there is no $\langle W, N, \overline{N}, h \rangle_m$ such that $h_m(\Gamma_1) = 1$ but $h_{m(s)}(/A/) = 0$ $\llbracket m(s) \notin /N/ \rrbracket$. So by V$\upsilon$X*, $\Gamma_1 \models^*_{\upsilon X} /A/_s$ $\llbracket /n/[s] \rrbracket$, where this is just to say, $\Gamma_1 \models^*_{\upsilon X} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^*_{\upsilon X} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, ∧I, ∧E, ∨I, ∨E, ¬I, ¬E or, depending on the system, →I4, →E4, →I*, →E*, K, NI, Ca, or Cb. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

(∧I)

(∧E)

(∨I)

(∨E)

(¬I) If $\mathcal{P}_k$ arises by ¬I, then the picture is like this,

$$\begin{array}{r|l}
 & \begin{array}{|l} \underline{/A/_s} \\ \\ \end{array} \\
i & \begin{array}{|l} /\!\!/B/\!\!/_t \end{array} \\
j & \begin{array}{|l} \backslash\!\!\backslash\neg B\backslash\!\!\backslash_t \end{array} \\
k & \backslash\neg A\backslash_s
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\backslash\neg A\backslash_s$. By assumption, $\Gamma_i \models^*_{\upsilon X} /\!\!/B/\!\!/_t$ and $\Gamma_j \models^*_{\upsilon X} \backslash\!\!\backslash\neg B\backslash\!\!\backslash_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{/A/_s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{/A/_s\}$; so by L7.1, $\Gamma_k \cup \{/A/_s\} \models^*_{\upsilon X} /\!\!/B/\!\!/_t$ and $\Gamma_k \cup \{/A/_s\} \models^*_{\upsilon X} \backslash\!\!\backslash\neg B\backslash\!\!\backslash_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} \backslash\neg A\backslash_s$; then by V$\upsilon$X*, there is an υX interpretation $\langle W, N, \overline{N}, h \rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\backslash\neg A\backslash) = 0$; so by H$\upsilon$(¬), $h_{m(s)}(/A/) = 1$; so $h_m(\Gamma_k) = 1$ and

$h_{m(s)}(/A/) = 1$; so $h_m(\Gamma_k \cup \{/A/_s\}) = 1$; so by V$\upsilon$X*, $h_{m(t)}(/\!\!/B/\!\!/) = 1$ and $h_{m(t)}(\backslash\!\backslash\neg B\backslash\!\backslash) = 1$; from the latter, by H$\upsilon(\neg)$, $h_{m(t)}(/\!\!/B/\!\!/) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} \backslash\neg A\backslash_s$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

($\neg$E)

($\to$I4) If $\mathcal{P}_k$ arises by $\to$I4, then the picture is like this,

$$\begin{array}{r|l}
i & /n/[s] \\
 & \quad\vert\ \underline{A_t} \\
 & \quad\vert \\
 & \quad\vert \\
j & \quad\vert\ /B/_t \\
k & /A \to B/_s
\end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $/A \to B/_s$. By assumption, $\Gamma_i \models^*_{\upsilon X} /n/[s]$ and $\Gamma_j \models^*_{\upsilon X} /B/_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_t\}$; so by L7.1, $\Gamma_k \models^*_{\upsilon X} /n/[s]$ and $\Gamma_k \cup \{A_t\} \models^*_{\upsilon X} /B/_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} /A \to B/_s$; then by V$\upsilon$X*, there is an υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/A \to B/) = 0$; since $h_m(\Gamma_k) = 1$, by VυX*, $m(s) \in /N/$; so, since $h_{m(s)}(/A \to B/) = 0$, by H$\upsilon(\to)_4$, there is some $w \in W$ such that $h_w(A) = 1$ and $h_w(/B/) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, N, \overline{N}, h\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$, $h_{m'(t)}(A) = 1$; so $h_{m'}(\Gamma_k \cup \{A_t\}) = 1$; so by V$\upsilon$X*, $h_{m'(t)}(/B/) = 1$. But $m'(t) = w$; so $h_w(/B/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} /A \to B/_s$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

($\to$E4) If $\mathcal{P}_k$ arises by $\to$E4, then the picture is like this,

$$\begin{array}{r|l}
h & /n/[s] \\
i & /A \to B/_s \\
j & A_t \\
 & \\
k & /B/_t
\end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $/B/_t$. By assumption, $\Gamma_h \models^*_{\upsilon X} /n/[s]$, $\Gamma_i \models^*_{\upsilon X} /A \to B/_s$, and $\Gamma_j \models^*_{\upsilon X} A_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$, and $\Gamma_j \subseteq \Gamma_k$; so by L7.1, $\Gamma_k \models^*_{\upsilon X} /n/[s]$, $\Gamma_k \models^*_{\upsilon X} /A \to B/_s$, and $\Gamma_k \models^*_{\upsilon X} A_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} /B/_t$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(/B/) =$

0; since $h_m(\Gamma_k) = 1$, by VυX*, $m(s) \in /N/$, $h_{m(s)}(/A \to B/) = 1$, and $h_{m(t)}(A) = 1$; from the second of these, since $m(s) \in /N/$, by H$\upsilon(\to)_4$, there is no $w \in W$ such that $h_w(A) = 1$ and $h_w(/B/) = 0$; so it is not the case that $h_{m(t)}(A) = 1$ and $h_{m(t)}(/B/) = 0$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{\upsilon X} /B/_t$, which is to say, $\Gamma_k \vDash^*_{\upsilon X} \mathcal{P}_k$.

($\to$I$*$) If $\mathcal{P}_k$ arises by $\to$I$*$, then the picture is like this,

$$
\begin{array}{r|l}
i & /n/[s] \\
 & \quad|\ \sslash A\sslash_t \\
 & \quad|\underline{\quad} \\
 & \quad| \\
j & \quad|\ \sslash B\sslash_t \\
k & /A \to B/_s
\end{array}
$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $/A \to B/_s$. By assumption, $\Gamma_i \vDash^*_{\upsilon X} /n/[s]$ and $\Gamma_j \vDash^*_{\upsilon X} \sslash B\sslash_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{\sslash A\sslash_t\}$; so by L7.1, $\Gamma_k \vDash^*_{\upsilon X} /n/[s]$ and $\Gamma_k \cup \{\sslash A\sslash_t\} \vDash^*_{\upsilon X} \sslash B\sslash_t$. Suppose $\Gamma_k \nvDash^*_{\upsilon X} /A \to B/_s$; then by V$\upsilon$X*, there is an υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/A \to B/) = 0$; since $h_m(\Gamma_k) = 1$, by VυX*, $m(s) \in /N/$; so, since $h_{m(s)}(/A \to B/) = 0$, by H$\upsilon(\to)_*$, there is some $x \in W$ such that $h_x(A) = 1$ and $h_x(B) = 0$, or $h_x(\overline{A}) = 1$ and $h_x(\overline{B}) = 0$. Without loss of generality, suppose $h_x(A) = 1$ and $h_x(B) = 0$; then by L7.0, there is an interpretation $\langle W', N', \overline{N}', h'\rangle$ where $h'_{w'}(/P/) = 1$ iff $h_w(/P/) = 1$ and $h'_{w^*}(/P/) = 1$ iff $h_w(\backslash P\backslash) = 1$. So with $m(s) = w$ iff $m'(s) = w'$, it remains that $h'_{m'}(\Gamma_k) = 1$; and we have that $x', x^* \in W'$ are such that $h'_{x'}(A) = 1$ and $h'_{x'}(B) = 0$, and $h'_{x^*}(\overline{A}) = 1$ and $h'_{x^*}(\overline{B}) = 0$; one of these is a y such that $h'_y(\sslash A\sslash) = 1$ and $h'_y(\sslash B\sslash) = 0$. Now consider a map n like m' except that $n(t) = y$, and consider $\langle W', N', \overline{N}', h'\rangle_n$; since t does not appear in Γ_k, it remains that $h'_n(\Gamma_k) = 1$; and since $n(t) = y$, $h'_{n(t)}(\sslash A\sslash) = 1$; so $h'_n(\Gamma_k \cup \{\sslash A\sslash_t\}) = 1$; so by V$\upsilon$X*, $h'_{n(t)}(\sslash B\sslash) = 1$. But $n(t) = y$; so $h'_y(\sslash B\sslash) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{\upsilon X} /A \to B/_s$, which is to say, $\Gamma_k \vDash^*_{\upsilon X} \mathcal{P}_k$.

($\to$E$*$) If $\mathcal{P}_k$ arises by $\to$E$*$, then the picture is like this,

$$\begin{array}{r|l} h & /n/[s] \\ i & /A \to B/_s \\ j & /\!/A/\!/_t \\ & \\ k & /\!/B/\!/_t \end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $/\!/B/\!/_t$. By assumption, $\Gamma_h \models^*_{\upsilon X} /n/[s]$, $\Gamma_i \models^*_{\upsilon X} /A \to B/_s$, and $\Gamma_j \models^*_{\upsilon X} /\!/A/\!/_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$, and $\Gamma_j \subseteq \Gamma_k$; so by L7.1, $\Gamma_k \models^*_{\upsilon X} /n/[s]$, $\Gamma_k \models^*_{\upsilon X} /A \to B/_s$, and $\Gamma_k \models^*_{\upsilon X} /\!/A/\!/_t$. Suppose $\Gamma_k \not\models^*_{\upsilon X} /\!/B/\!/_t$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(/\!/B/\!/) = 0$; since $h_m(\Gamma_k) = 1$, by VυX*, $m(s) \in /N/$, $h_{m(s)}(/A \to B/) = 1$, and $h_{m(t)}(/\!/A/\!/) = 1$; from the second of these, since $m(s) \in /N/$, by H$\upsilon(\to)_*$, there is no $w \in W$ such that $h_w(/\!/A/\!/) = 1$ and $h_w(/\!/B/\!/) = 0$; so it is not the case that $h_{m(t)}(/\!/A/\!/) = 1$ and $h_{m(t)}(/\!/B/\!/) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} /\!/B/\!/_t$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

(K) If $\mathcal{P}_k$ arises by K, then the picture is like this,

$$\begin{array}{r|l} & \\ & \\ k & /n/[s] \end{array}$$

where $\mathcal{P}_k$ is $/n/[s]$. Where this rule is in $N\upsilon$X, υX includes condition K. Suppose $\Gamma_k \not\models^*_{\upsilon X} /n/[s]$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $m(s) \notin /N/$. But by condition K, $N = \overline{N} = W$; so $m(s) \in /N/$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} /n/[s]$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

(NI) If $\mathcal{P}_k$ arises by NI, then the picture is like this,

$$\begin{array}{r|l} & \\ & \\ k & n[0] \end{array}$$

where $\mathcal{P}_k$ is $n[0]$. Suppose $\Gamma_k \not\models^*_{\upsilon X} n[0]$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $m(0) \notin N$. But by construction, $m(0) \in N$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} n[0]$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

(Ca) If $\mathcal{P}_k$ arises by Ca then the picture is like this,

$$\begin{array}{r|l} i & /n/[s] \\ & \\ k & \backslash n\backslash[s] \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $\backslash n\backslash[s]$. Where this rule is in $N\upsilon$X, υX includes condition 4. By assumption, $\Gamma_i \models^*_{\upsilon X} /n/[s]$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L7.1, $\Gamma_k \models^*_{\upsilon X} /n/[s]$. Suppose $\Gamma_k \not\models^*_{\upsilon X} \backslash n\backslash[s]$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $m(s) \notin \backslash N\backslash$; since $h_m(\Gamma_k) = 1$, by VυX*, $m(s) \in /N/$. But by condition 4, $N = \overline{N}$; so $m(s) \in \backslash N\backslash$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} \backslash n\backslash[s]$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

(Cb) If $\mathcal{P}_k$ arises by Cb then the picture is like this,

$$\begin{array}{r|l} i & /n/[a] \\ j & /\!/A/\!/_a \\ & \\ k & \backslash\!\backslash A\backslash\!\backslash_a \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\backslash\!\backslash A\backslash\!\backslash_a$. By assumption, $\Gamma_i \models^*_{\upsilon X} /n/[a]$ and $\Gamma_j \models^*_{\upsilon X} /\!/A/\!/_a$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L7.1, $\Gamma_k \models^*_{\upsilon X} /n/[a]$ and $\Gamma_k \models^*_{\upsilon X} /\!/A/\!/_a$. Suppose $\Gamma_k \not\models^*_{\upsilon X} \backslash\!\backslash A\backslash\!\backslash_a$; then by V$\upsilon$X*, there is some υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(a)}(\backslash\!\backslash A\backslash\!\backslash) = 0$; since $h_m(\Gamma_k) = 1$, by VυX*, $m(a) \in /N/$ and $h_{m(a)}(/\!/A/\!/) = 1$.

Now, by induction on the number of operators in $/\!/A/\!/$, we show that if $x \in /N/$, then $h_x(/\!/A/\!/) = 1$ iff $h_x(\backslash\!\backslash A\backslash\!\backslash) = 1$. Suppose $x \in /N/$.

Basis: Suppose $/\!/A/\!/$ is a parameter p. By requirement CL, $h_x(/\!/p/\!/) = 1$ iff $h_x(\backslash\!\backslash p\backslash\!\backslash) = 1$; so $h_x(/\!/A/\!/) = 1$ iff $h_x(\backslash\!\backslash A\backslash\!\backslash) = 1$.

Assp: For $0 \leq i < k$, if $/\!/A/\!/$ has i operators, then $h_x(/\!/A/\!/) = 1$ iff $h_x(\backslash\!\backslash A\backslash\!\backslash) = 1$.

Show: If $/\!/A/\!/$ has k operators, $h_x(/\!/A/\!/) = 1$ iff $h_x(\backslash\!\backslash A\backslash\!\backslash) = 1$

If $/\!/A/\!/$ has k operators then it is of the form, $/\!/\neg P/\!/$, $/\!/P \wedge Q/\!/$, $/\!/P \vee Q/\!/$, or $/\!/P \rightarrow Q/\!/$, where P and Q have $< k$ operators.

($\neg$) Suppose $/\!/A/\!/$ is $/\!/\neg P/\!/$. Then $h_x(/\!/A/\!/) = 1$ iff $h_x(/\!/\neg P/\!/) = 1$; by Hυ($\neg$), iff $h_x(\backslash\!\backslash P\backslash\!\backslash) = 0$; by assumption, iff $h_x(/\!/Q/\!/) = 0$; by Hυ($\neg$), iff $h_x(\backslash\!\backslash \neg P\backslash\!\backslash) = 1$; iff $h_x(\backslash\!\backslash A\backslash\!\backslash) = 1$.

($\wedge$) Suppose $/\!/A/\!/$ is $/\!/P \wedge Q/\!/$. Then $h_x(/\!/A/\!/) = 1$ iff $h_x(/\!/P \wedge Q/\!/) = 1$; by H$\upsilon$($\wedge$), iff $h_x(/\!/P/\!/) = 1$ and $h_x(/\!/Q/\!/) = 1$; by assumption, iff $h_x(\backslash\!\backslash P\backslash\!\backslash) = 1$ and $h_x(\backslash\!\backslash Q\backslash\!\backslash) = 1$; by H$\upsilon$($\wedge$), iff $h_x(\backslash\!\backslash P \wedge Q\backslash\!\backslash) = 1$; iff $h_x(\backslash\!\backslash A\backslash\!\backslash) = 1$.

($\vee$)

$(\rightarrow)_4$ Suppose $\sslash A\sslash$ is $\sslash P \rightarrow Q\sslash$. Then $h_x(\sslash A\sslash) = 1$ iff $h_x(\sslash P \wedge Q\sslash) = 1$; since $x \in /N/$, by H$\upsilon(\rightarrow)_4$, iff there is no $y \in W$ such that $h_y(A) = 1$ and $h_y(\sslash B\sslash) = 0$; by assumption, iff there is no $y \in W$ such that $h_y(A) = 1$ and $h_y(\bbslash B\bbslash) = 0$; by H$\upsilon(\rightarrow)_4$, iff $h_x(\bbslash P \wedge Q\bbslash) = 1$.

$(\rightarrow)_*$ Suppose $\sslash A\sslash$ is $\sslash P \rightarrow Q\sslash$. Then $h_x(\sslash A\sslash) = 1$ iff $h_x(\sslash P \wedge Q\sslash) = 1$; since $x \in /N/$, by H$\upsilon(\rightarrow)_*$, iff there is no $y \in W$ such that $h_y(\sslash A\sslash) = 1$ and $h_y(\sslash B\sslash) = 0$; by assumption, iff there is no $y \in W$ such that $h_y(\bbslash A\bbslash) = 1$ and $h_y(\bbslash B\bbslash) = 0$; by H$\upsilon(\rightarrow)_4$, iff $h_x(\bbslash P \wedge Q\bbslash) = 1$.

For any such $\sslash A\sslash$, $h_x(\sslash A\sslash) = 1$ iff $h_x(\bbslash A\bbslash) = 1$.

So, returning to the case for (Cb), $h_{m(a)}(\bbslash A\bbslash) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\upsilon X} \bbslash A\bbslash_a$, which is to say, $\Gamma_k \models^*_{\upsilon X} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{\upsilon X} \mathcal{P}_i$.

THEOREM 7.2 *$N\upsilon X$ is complete: if $\Gamma \models_{\upsilon X} A$ then $\Gamma \vdash_{N\upsilon X} A$.*

Suppose $\Gamma \models_{\upsilon X} A$; then $\Gamma_0 \models^*_{\upsilon X} A_0$; we show that $\Gamma_0 \vdash^*_{N\upsilon X} A_0$. As usual, this reduces to the standard notion. For the following, fix on some particular υX. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{N\upsilon X} /A/_s$ and $\Gamma \vdash^*_{N\upsilon X} \backslash\neg A\backslash_s$.

L7.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{N\upsilon X} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

Suppose s is 0 or appears in Γ and $\Gamma \nvdash^*_{N\upsilon X} \backslash\neg P\backslash_s$ but $\Gamma \cup \{/P/_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{/P/_s\} \vdash^*_{N\upsilon X} \sslash A\sslash_t$ and $\Gamma \cup \{/P/_s\} \vdash^*_{N\upsilon X} \bbslash\neg A\bbslash_t$. But then we can argue,

1	Γ	
2	$/P/_s$	A (c, $\neg$I)
3	$\sslash A\sslash_t$	from $\Gamma \cup \{/P/_s\}$
4	$\bbslash\neg A\bbslash_t$	from $\Gamma \cup \{/P/_s\}$
5	$\backslash\neg P\backslash_s$	2-4 $\neg$I

where the assumption is allowed insofar as s is either 0 or appears in Γ; so $\Gamma \vdash^*_{N\upsilon X} \backslash\neg P\backslash_s$. But this is impossible; reject the assumption: if s is 0 or appears in Γ and $\Gamma \nvdash^*_{N\upsilon X} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

L7.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$

Proof by construction as usual.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^{*}_{N\upsilon X} /A/_s$ or $\Gamma \vdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$.

SGT Γ is a SCAPEGOAT set for $(\to)_{K_4}$ iff for every formula of the form $\neg(A \to B)$, if $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg(A \to B)/_s$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_4} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg B/_t$.

Γ is a SCAPEGOAT set for $(\to)_{N_4}$ iff for every formula of the form $\neg(A \to B)$, if $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg(A \to B)/_0$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_4} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_4} /\neg B/_t$.

Γ is a SCAPEGOAT set for $(\to)_{K_*}$ iff for every formula of the form $\neg(A \to B)$, if $\Gamma \vdash^{*}_{N\upsilon K_*} /\neg(A \to B)/_s$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_*} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_*} \overline{\neg B}_t$.

Γ is a SCAPEGOAT set for $(\to)_{N_*}$ iff for every formula of the form $\neg(A \to B)$, if $\Gamma \vdash^{*}_{N\upsilon K_*} \overline{\neg(A \to B)}_0$ then there is some t such that $\Gamma \vdash^{*}_{N\upsilon K_*} A_t$ and $\Gamma \vdash^{*}_{N\upsilon K_*} \overline{\neg B}_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L7.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the formulas; let $\mathcal{E}_0$ be this enumeration. Then for the first $/A/_s$ in $\mathcal{E}_{i-1}$ such that s is 0 or included in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $/A/_s$, and set,

$$\begin{array}{ll} \Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \vdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s \\ \Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\} & \text{if } \Omega_{i-1} \nvdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s \end{array}$$

and

$$\begin{array}{lll} \upsilon K_4: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } /\neg(P \to Q)/_s \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, /\neg Q/_t\} & \text{if } A_s \text{ is of the form } /\neg(P \to Q)/_s \\ \upsilon N_4: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } /\neg(P \to Q)/_0 \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, /\neg Q/_t\} & \text{if } A_s \text{ is of the form } /\neg(P \to Q)/_0 \\ \upsilon K_*: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } /\neg(P \to Q)/_s \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, \overline{\neg Q}_t\} & \text{if } A_s \text{ is of the form } /\neg(P \to Q)/_s \\ \upsilon N_*: & \Omega_i = \Omega_{i^*} & \text{if } A_s \text{ is not of the form } \overline{\neg(P \to Q)}_0 \\ & \Omega_i = \Omega_{i^*} \cup \{P_t, \overline{\neg Q}_t\} & \text{if } A_s \text{ is of the form } \overline{\neg(P \to Q)}_0 \end{array}$$

-where t is the first subscript not included in Ω_{i^*}

then

$$\Gamma' = \textstyle\bigcup_{i \geq 0} \Omega_i$$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being

0. Suppose s appears in Γ'; then there is some Ω_i in which it is first appears; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L7.4 For any s included in Γ', Γ' is s-maximal.

Suppose s is included in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^{*}_{N\upsilon X} /A/_s$ and $\Gamma' \nvdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$ then $\Gamma' \vdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$; but $\Gamma' \nvdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$; so $\Omega_{i-1} \nvdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$; so since s is included in Γ', there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\}$; so by construction, $/A/_s \in \Gamma'$; so $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L7.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$, (iii) $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\}$ in υK_4 or υN_4, or (iv) $\Omega_{k^*} \cup \{P_t, \overline{\neg Q}_t\}$ in υK_* or υN_*.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^{*}_{N\upsilon X} \backslash\neg A\backslash_s$; so by L7.2, $\Omega_{k-1} \cup \{/A/_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\}$ in υK_4 or υN_4. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg(P \rightarrow Q)/_s \in \Omega_{k^*}$ (in υN_4, with $s = 0$). Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\} \vdash^{*}_{N\upsilon X} /\!\!/A/\!\!/_u$ and $\Omega_{k^*} \cup \{P_t, /\neg Q/_t\} \vdash^{*}_{N\upsilon X} \backslash\!\!\backslash\neg A\backslash\!\!\backslash_u$. So reason as follows,

$$\begin{array}{r|ll}
1 & \Omega_{k^*} & \\
2 & \vert\; \underline{P_t} & \text{A } (g, \to\text{I}_4) \\
3 & \vert\;\vert\; \underline{/\neg Q/_t} & \text{A } (c, \neg\text{E}) \\
4 & \vert\;\vert\; /\!/A/\!/_u & \text{from } \Omega_{k^*} \cup \{P_t, /\neg Q/_t\} \\
5 & \vert\;\vert\; \backslash\!\backslash\neg A\backslash\!\backslash_u & \text{from } \Omega_{k^*} \cup \{P_t, /\neg Q/_t\} \\
6 & \vert\; \backslash Q\backslash_t & \text{3-5 } \neg\text{E} \\
7 & \backslash P \to Q\backslash_s & \text{2-6 } \to\text{I}_4
\end{array}$$

where, by construction, t is not in Ω_{k^*} and for υN_4, $s = 0$. So $\Omega_{k^*} \vdash^*_{N\upsilon X} \backslash P \to Q\backslash_s$; but $/\neg(P \to Q)/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{N\upsilon X} /\neg(P \to Q)/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\}$ in υK_* or υN_*. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg(P \to Q)/_s \in \Omega_{k^*}$ (in υK_*, with overline and $s = 0$). Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\} \vdash^*_{N\upsilon X} /\!/A/\!/_u$ and $\Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\} \vdash^*_{N\upsilon X} \backslash\!\backslash\neg A\backslash\!\backslash_u$. So reason as follows,

$$\begin{array}{r|ll}
1 & \Omega_{k^*} & \\
2 & \vert\; \underline{P_t} & \text{A } (g, \to\text{I}*) \\
3 & \vert\;\vert\; \underline{\overline{\neg Q_t}} & \text{A } (c, \neg\text{E}) \\
4 & \vert\;\vert\; /\!/A/\!/_u & \text{from } \Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\} \\
5 & \vert\;\vert\; \backslash\!\backslash\neg A\backslash\!\backslash_u & \text{from } \Omega_{k^*} \cup \{P_t, \overline{\neg Q_t}\} \\
6 & \vert\; Q_t & \text{3-5 } \neg\text{E} \\
7 & \backslash P \to Q\backslash_s & \text{2-6 } \to\text{I}*
\end{array}$$

where, by construction, t is not in Ω_{k^*} and for υN_*, $\backslash P \to Q\backslash_s$ is without overline and $s = 0$. So $\Omega_{k^*} \vdash^*_{N\upsilon X} \backslash P \to Q\backslash_s$; but $/\neg(P \to Q)/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{N\upsilon X} /\neg(P \to Q)/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L7.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 and L6.6.

L7.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $(\to)_{K_4}$, $(\to)_{N_4}$, $(\to)_{K_*}$, and $(\to)_{N_*}$.

For $(\to)_{K_4}$ and $(\to)_{N_4}$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{N\upsilon X} /\neg(P \to Q)/_s$. By L7.6, Γ' is consistent; and by the constraints on subscripts,

s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{N\upsilon X} \backslash\neg\neg(P \to Q)\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg(P \to Q)/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_t, /\neg Q/_t\}$; so by construction, $P_t \in \Gamma'$ and $/\neg Q/_t \in \Gamma'$; so $\Gamma' \vdash^*_{N\upsilon X} P_t$ and $\Gamma' \vdash^*_{N\upsilon X} /\neg Q/_t$. So Γ' is a scapegoat set for $(\to)_{K_4}$ and $(\to)_{N_4}$ – where the argument for $(\to)_{N_4}$ assumes $s = 0$.

For $(\to)_{K_*}$ and $(\to)_{N_*}$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{N\upsilon X} /\neg(P \to Q)/_s$. By L7.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{N\upsilon X} \backslash\neg\neg(P \to Q)\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg(P \to Q)/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_t, \overline{\neg Q_t}\}$; so by construction, $P_t \in \Gamma'$ and $\overline{\neg Q_t} \in \Gamma'$; so $\Gamma' \vdash^*_{N\upsilon X} P_t$ and $\Gamma' \vdash^*_{N\upsilon X} \overline{\neg Q_t}$. So Γ' is a scapegoat set for $(\to)_{K_*}$ and $(\to)_{N_*}$ – where the argument for $(\to)_{N_*}$ assumes $/\neg(P \to Q)/_s$ is with overline and $s = 0$.

C(I) We construct an interpretation $I = \langle W, N, \overline{N}, h\rangle$ based on Γ' as follows.

υK_x: For the K systems, let W have a member w_s corresponding to each subscript s included in Γ'. Then set $N = \overline{N} = W$ and $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /p/_s$.

υN_4: Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $N = \overline{N} = \{w_0\}$; $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /p/_s$; and for $s \neq 0$, $h_{w_s}(/A \to B/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A \to B/_s$.

υN_*: Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $N = \{w_0\}$ and $\overline{N} = \phi$; $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /p/_s$; $h_{w_s}(\overline{P \to Q}) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} \overline{(P \to Q)}_s$; and for $s \neq 0$, $h_{w_s}(A \to B) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} (A \to B)_s$.

L7.8 If Γ_0 is consistent then for $\langle W, N, \overline{N}, h\rangle$ constructed as above, and for any s included in Γ', $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L7.4, Γ' is s-maximal. By L7.6 and L7.7, Γ' is consistent and a scapegoat set for the different conditionals. Now by induction on the number of operators in $/A/_s$,

Basis: If $/A/_s$ has no operators, then it is a parameter $/p/_s$ and by construction, $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /p/_s$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

Assp: For any i, $0 \leq i < k$, if $/A/_s$ has i operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

Show: If $/A/_s$ has k operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

If $/A/_s$ has k operators, then it is of the form $/\neg P/_s$, $/P \wedge Q/_s$, $/P \vee Q/_s$ or $/P \rightarrow Q/_s$, where P and Q have $< k$ operators.

($\neg$) $/A/_s$ is $/\neg P/_s$. (i) Suppose $h_{w_s}(/A/) = 1$; then $h_{w_s}(/\neg P/) = 1$; so by H$\upsilon(\neg)$, $h_{w_s}(\backslash P\backslash) = 0$; so by assumption, $\Gamma' \nvdash^*_{N\upsilon X} \backslash P\backslash_s$; so by s-maximality, $\Gamma' \vdash^*_{N\upsilon X} /\neg P/_s$, where this is to say, $\Gamma' \vdash^*_{N\upsilon X} /A/_s$. (ii) Suppose $\Gamma' \vdash^*_{N\upsilon X} /A/_s$; then $\Gamma' \vdash^*_{N\upsilon X} /\neg P/_s$; so by consistency, $\Gamma' \nvdash^*_{N\upsilon X} \backslash P\backslash_s$; so by assumption, $h_{w_s}(\backslash P\backslash) = 0$; so by H$\upsilon(\neg)$, $h_{w_s}(/\neg P/) = 1$, where this is to say, $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

($\wedge$)

($\vee$)

($\rightarrow$) $/A/_s$ is $/P \rightarrow Q/_s$. (i) Suppose $h_{w_s}(/A/) = 1$ but $\Gamma' \nvdash^*_{N\upsilon X} /A/_s$; then $h_{w_s}(/P \rightarrow Q/) = 1$, but $\Gamma' \nvdash^*_{N\upsilon X} /P \rightarrow Q/_s$; from the latter, by s-maximality, $\Gamma' \vdash^*_{N\upsilon X} \backslash\neg(P \rightarrow Q)\backslash_s$.

υK_4: In this case, $N = \overline{N} = K$; so $w_s \in /N/$. Since Γ' is a scapegoat set for $(\rightarrow)_{K_4}$, there is some t such that $\Gamma' \vdash^*_{N\upsilon K_4} P_t$ and $\Gamma' \vdash^*_{N\upsilon K_4} \backslash\neg Q\backslash_t$; from the latter, by consistency, $\Gamma' \nvdash^*_{N\upsilon K_4} /Q/_t$; so by assumption, $h_{w_t}(P) = 1$ and $h_{w_t}(/Q/) = 0$; so since $w_s \in /N/$, by H$\upsilon(\rightarrow)_4$, $h_{w_s}(/P \rightarrow Q/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

υN_4: In this case, when $s = 0$, $w_s \in /N/$ and reasoning is as immediately above. Otherwise, by construction, if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

υK_*: In this case, $N = \overline{N} = K$; so $w_s \in /N/$. Since Γ' is a scapegoat set for $(\rightarrow)_{K_*}$, there is some t such that $\Gamma' \vdash^*_{N\upsilon K_*} P_t$ and $\Gamma' \vdash^*_{N\upsilon K_*} \overline{\neg Q}_t$; from the latter, by consistency, $\Gamma' \nvdash^*_{N\upsilon K_*} Q_t$; so by assumption, $h_{w_t}(P) = 1$ and $h_{w_t}(Q) = 0$; so since $w_s \in /N/$, by H$\upsilon(\rightarrow)_*$, $h_{w_s}(/P \rightarrow Q/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

υN_*: In this case, when $s = 0$ and $/P \rightarrow Q/$ is without overline – so that $\backslash\neg(P \rightarrow Q)\backslash$ is $\overline{\neg(P \rightarrow Q)}$ – $w_s \in /N/$ and reasoning is as immediately above. Otherwise, by construction, if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

So in any of these cases, if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$.

(ii) Suppose $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$ but $h_{w_s}(/A/) = 0$; then $\Gamma' \vdash^{*}_{N\upsilon X} /P \rightarrow Q/_s$ but $h_{w_s}(/P \rightarrow Q/) = 0$.

υK_4: From the latter, by H$\upsilon(\rightarrow)_4$, there is some $w_t \in W$ such that $h_{w_t}(P) = 1$ and $h_{w_t}(/Q/) = 0$; so by assumption, $\Gamma' \vdash^{*}_{N\upsilon K_4} P_t$ and $\Gamma' \nvdash^{*}_{N\upsilon K_4} /Q/_t$; so by s-maximality, $\Gamma' \vdash^{*}_{N\upsilon K_4} \backslash\neg Q\backslash_t$. So by reasoning as follows,

$$\begin{array}{r|l|ll}
1 & \multicolumn{3}{l}{\Gamma'} \\
2 & & \underline{/P \rightarrow Q/_s} & \text{A } (c, \neg\text{I}) \\
3 & & P_t & \text{from } \Gamma' \\
4 & & /Q/_t & 2{,}3 \rightarrow\text{E}4 \\
5 & & \backslash\neg Q\backslash_t & \text{from } \Gamma' \\
6 & \multicolumn{2}{l}{\backslash\neg(P \rightarrow Q)\backslash_s} & 2\text{-}5\ \neg\text{I}
\end{array}$$

$\Gamma' \vdash^{*}_{N\upsilon K_4} \backslash\neg(P \rightarrow Q)\backslash_s$; so by consistency, $\Gamma' \nvdash^{*}_{N\upsilon K_4} /P \rightarrow Q/_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$.

υN_4: When $s = 0$, the reasoning is as immediately above. Otherwise, by construction, if $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$, then $h_{w_s}(/A/) = 1$.

υK_*: From the latter, by H$\upsilon(\rightarrow)_*$, there is some $w_t \in W$ such that $h_{w_t}(/\!\!/P/\!\!/) = 1$ and $h_{w_t}(/\!\!/Q/\!\!/) = 0$; so by assumption, $\Gamma' \vdash^{*}_{N\upsilon K_4} /\!\!/P/\!\!/_t$ and $\Gamma' \nvdash^{*}_{N\upsilon K_4} /\!\!/Q/\!\!/_t$; so by s-maximality, $\Gamma' \vdash^{*}_{N\upsilon K_4} \backslash\!\!\backslash\neg Q\backslash\!\!\backslash_t$. So by reasoning as follows,

$$\begin{array}{r|l|ll}
1 & \multicolumn{3}{l}{\Gamma'} \\
2 & & \underline{/P \rightarrow Q/_s} & \text{A } (c, \neg\text{I}) \\
3 & & /\!\!/P/\!\!/_t & \text{from } \Gamma' \\
4 & & /\!\!/Q/\!\!/_t & 2{,}3 \rightarrow\text{E}* \\
5 & & \backslash\!\!\backslash\neg Q\backslash\!\!\backslash_t & \text{from } \Gamma' \\
6 & \multicolumn{2}{l}{\backslash\neg(P \rightarrow Q)\backslash_s} & 2\text{-}5\ \neg\text{I}
\end{array}$$

$\Gamma' \vdash^{*}_{N\upsilon K_*} \backslash\neg(P \rightarrow Q)\backslash_s$; so by consistency, $\Gamma' \nvdash^{*}_{N\upsilon K_*} /P \rightarrow Q/_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$.

υN_*: When $s = 0$ and $/P \rightarrow Q/$ is without overline, the reasoning is as immediately above. Otherwise, by construction, if $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$.

So in any of these cases, if $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$ then $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^{*}_{N\upsilon X} /A/_s$.

For any A_s, $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{N\upsilon X} /A/_s$.

L7.9 If Γ_0 is consistent, then $\langle W, N, \overline{N}, h\rangle$ constructed as above is an υX interpretation.

This is immediate, by construction.

Map For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L7.10 If Γ_0 is consistent, then $h_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 and L6.9.

Main result: Suppose $\Gamma \vDash_{\upsilon X} A$ but $\Gamma \nvdash_{N\upsilon X} A$. Then $\Gamma_0 \vDash^*_{\upsilon X} A_0$ but $\Gamma_0 \nvdash^*_{N\upsilon X} A_0$. By (DN), if $\Gamma_0 \vdash^*_{N\upsilon X} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{N\upsilon X} A_0$; so $\Gamma_0 \nvdash^*_{N\upsilon X} \neg\neg A_0$; so by L7.2, $\Gamma_0 \cup \{\overline{\neg A_0}\}$ is consistent; so by L7.9 and L7.10, there is an υX interpretation $\langle W, N, \overline{N}, h\rangle_m$ constructed as above such that $h_m(\Gamma_0 \cup \{\overline{\neg A_0}\}) = 1$; so $h_{m(0)}(\overline{\neg A}) = 1$; so by H$\upsilon(\neg)$, $h_{m(0)}(A) = 0$; so $h_m(\Gamma_0) = 1$ and $h_{m(0)}(A) = 0$; so by VυX*, $\Gamma_0 \nvDash^*_{\upsilon X} A_0$. This is impossible; reject the assumption: if $\Gamma \vDash_{\upsilon X} A$, then $\Gamma \vdash_{N\upsilon X} A$.

7.4 Soundness and Completeness: Ix

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, R, h\rangle$, let m be a map from subscripts into W. Then say $\langle W, R, h\rangle_m$ is $\langle W, R, h\rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $h_m(\Gamma) = 1$ iff for each $/A_s/ \in \Gamma$, $h_{m(s)}(/A/) = 1$, and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity for subscripts, overlines, and alternate expressions as indicated in double brackets as follows,

VIx* $\Gamma \vDash^*_{Ix} /A/_s$ $\llbracket s.t \rrbracket$ iff there is no Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma) = 1$ but $h_{m(s)}(/A/) = 0$ $\llbracket \langle m(s), m(t)\rangle \notin R \rrbracket$.

NIx* $\Gamma \vdash^*_{NIx} /A/_s$ $\llbracket s.t \rrbracket$ iff there is an NIx derivation of $/A/_s$ $\llbracket s.t \rrbracket$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A are without overlines and have subscript 0 (and so do not include expressions of the sort $s.t$). As usual, for the following, cases omitted are like ones worked, and so left to the reader.

Theorem 7.3 *NIx is sound: If* $\Gamma \vdash_{NIx} A$ *then* $\Gamma \vDash_{Ix} A$.

L7.1*a* If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^{*}_{Ix} /P/_s \llbracket s.t \rrbracket$ then $\Gamma' \models^{*}_{Ix} /P/_s \llbracket s.t \rrbracket$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \models^{*}_{Ix} /P/_s \llbracket s.t \rrbracket$, but $\Gamma' \not\models^{*}_{Ix} /P/_s \llbracket s.t \rrbracket$. From the latter, by VIx*, there is some Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma') = 1$ but $h_{m(s)}(/P/) = 0 \llbracket \langle m(s), m(t)\rangle \notin R \rrbracket$. But since $h_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $h_m(\Gamma) = 1$; so $h_m(\Gamma) = 1$ but $h_{m(s)}(/P/) = 0$ $\llbracket \langle m(s), m(t)\rangle \notin R \rrbracket$; so by VIx*, $\Gamma \not\models^{*}_{Ix} /P/_s \llbracket s.t \rrbracket$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models^{*}_{Ix} /P/_s \llbracket s.t \rrbracket$, then $\Gamma' \models^{*}_{Ix} /P/_s$ $\llbracket s.t \rrbracket$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^{*}_{NIx} \mathcal{P}$ then $\Gamma \models^{*}_{Ix} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are without overlines and have subscript 0. Suppose $\Gamma \vdash^{*}_{NIx} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^{*}_{Ix} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/_s \llbracket s.t \rrbracket$. Then $\Gamma_1 = \{/A/_s\}$ $\llbracket \{s.t\} \rrbracket$; so for any $\langle W, R, h\rangle_m$, $h_m(\Gamma_1) = 1$ iff $h_{m(s)}(/A/) = 1 \llbracket \langle m(s), m(t)\rangle \in R \rrbracket$; so there is no $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_1) = 1$ but $h_{m(s)}(/A/) = 0$ $\llbracket \langle m(s), m(t)\rangle \notin R \rrbracket$. So by VIx*, $\Gamma_1 \models^{*}_{Ix} /A/_s \llbracket s.t \rrbracket$, where this is just to say, $\Gamma_1 \models^{*}_{Ix} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^{*}_{Ix} \mathcal{P}_i$.

Show: $\Gamma_k \models^{*}_{Ix} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\neg$I, $\neg$E or, depending on the system, $\sqsupset$I, $\sqsupset$E, AMρ, AMτ, H$_{\text{I}}$, D, $\overline{\sqsupset}$I, $\overline{\sqsupset}$E, $\overline{\sqsupset}\text{I}_W$, or $\overline{\sqsupset}\text{E}_W$. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^{*}_{Ix} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\wedge$I)

($\wedge$E)

($\vee$I)

(∨E)

(¬I)

(¬E)

($\sqsupset$I) If $\mathcal{P}_k$ arises by $\sqsupset$I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & s.t \\
 & & A_t \\ \cline{3-3}
 & & \\
i & & B_t \\
k & \multicolumn{2}{l}{(A \sqsupset B)_s}
\end{array}$$

where $i < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $(A \sqsupset B)_s$. By assumption, $\Gamma_i \vDash^*_{Ix} B_t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t, A_t\}$; so by L7.1$a$, $\Gamma_k \cup \{s.t, A_t\} \vDash^*_{Ix} B_t$. Suppose $\Gamma_k \nvDash^*_{Ix} (A \sqsupset B)_s$; then by VIx*, there is an Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(A \sqsupset B) = 0$; so, by HIx($\sqsupset$), there is some $w \in W$ such that $m(s)Rw$ and $h_w(A) = 1$ but $h_w(B) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, h\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$, $m(s)Rm'(t)$ and $h_{m'(t)}(A) = 1$; so $h_{m'}(\Gamma_k \cup \{s.t, A_t\}) = 1$; so by VIx*, $h_{m'(t)}(B) = 1$. But $m'(t) = w$; so $h_w(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{Ix} (A \sqsupset B)_s$, which is to say, $\Gamma_k \vDash^*_{Ix} \mathcal{P}_k$.

($\sqsupset$E) If $\mathcal{P}_k$ arises by $\sqsupset$E, then the picture is like this,

$$\begin{array}{r|l}
h & (A \sqsupset B)_s \\
i & s.t \\
j & A_t \\
 & \\
k & B_t
\end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is B_t. By assumption, $\Gamma_h \vDash^*_{Ix} (A \sqsupset B)_s$, $\Gamma_i \vDash^*_{Ix} s.t$ and $\Gamma_j \vDash^*_{Ix} A_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L7.1$a$, $\Gamma_k \vDash^*_{Ix} (A \sqsupset B)_s$, $\Gamma_k \vDash^*_{Ix} s.t$ and $\Gamma_k \vDash^*_{Ix} A_t$. Suppose $\Gamma_k \nvDash^*_{Ix} B_t$; then by VIx*, there is some Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(B) = 0$; since $h_m(\Gamma_k) = 1$, by VIx*, $h_{m(s)}(A \sqsupset B) = 1$, $\langle m(s), m(t)\rangle \in R$ and $h_{m(t)}(A) = 1$; from the first of these, by HIx($\sqsupset$), there is no $w \in W$ such that $m(s)Rw$ and $h_w(A) = 1$ but $h_w(B) = 0$; so it is not the case that $\langle m(s), m(t)\rangle \in R$ and $h_{m(t)}(A) = 1$ but $h_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Ix} B_t$, which is to say, $\Gamma_k \models^*_{Ix} \mathcal{P}_k$.

(AMρ)

(AMτ)

(H$_{\mathrm{I}}$) If $\mathcal{P}_k$ arises by H$_{\mathrm{I}}$, then the picture is like this,

$$\begin{array}{r|l} i & A_s \\ j & s.t \\ & \\ k & A_t \end{array} \qquad \text{or} \qquad \begin{array}{r|l} i & \overline{A}_t \\ j & s.t \\ & \\ k & \overline{A}_s \end{array}$$

where $i, j < k$ and, in the left-hand case, $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{Ix} A_s$ and $\Gamma_j \models^*_{Ix} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L7.1*a*, $\Gamma_k \models^*_{Ix} A_s$ and $\Gamma_k \models^*_{Ix} s.t$. Suppose $\Gamma_k \not\models^*_{Ix} A_t$; then by VIx*, there is some *Ix* interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(A) = 0$; since $h_m(\Gamma_k) = 1$, by VIx*, $h_{m(s)}(A) = 1$ and $\langle m(s), m(t)\rangle \in R$.

Now, by induction on the number of operators in A, we show that if xRy, then if $h_x(A) = 1$, then $h_y(A) = 1$, and if $h_y(\overline{A}) = 1$, then $h_x(\overline{A}) = 1$. Suppose xRy.

Basis: Suppose A is a parameter p. (i) Suppose $h_x(A) = 1$; then $h_x(p) = 1$; so by condition h, $h_y(p) = 1$; so $h_y(A) = 1$.

(ii) Suppose $h_y(\overline{A}) = 1$; then $h_y(\overline{p}) = 1$; so by condition h, $h_x(\overline{p}) = 1$; so $h_x(\overline{A}) = 1$.

Assp: For $0 \leq i < k$, if A has i operators, then if $h_x(A) = 1$, then $h_y(A) = 1$, and if $h_y(\overline{A}) = 1$, then $h_x(\overline{A}) = 1$.

Show: If A has k operators, then if $h_x(A) = 1$, then $h_y(A) = 1$, and if $h_y(\overline{A}) = 1$, then $h_x(\overline{A}) = 1$.

If A has k operators then it is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \sqsupset Q$, where P and Q have $< k$ operators.

($\neg$) Suppose A is $\neg P$. (i) Suppose $h_x(A) = 1$; then $h_x(\neg P) = 1$; so by HIx($\neg$), $h_x(\overline{P}) = 0$; so by assumption, $h_y(\overline{P}) = 0$; so by HIx($\neg$), $h_y(\neg P) = 1$.

(ii) Suppose $h_y(\overline{A}) = 1$; then $h_y(\overline{\neg P}) = 1$; so by HIx($\neg$), $h_y(P) = 0$; so by assumption, $h_x(P) = 0$; so by HIx($\neg$), $h_x(\overline{\neg P}) = 1$.

($\wedge$) Suppose A is $P \wedge Q$. (i) Suppose $h_x(A) = 1$; then $h_x(P \wedge Q) = 1$; so by HIx($\wedge$), $h_x(P) = 1$ and $h_x(Q) = 1$; so by assumption, $h_y(P) = 1$ and $h_y(Q) = 1$; so by HIx($\wedge$), $h_y(P \wedge Q) = 1$; so $h_y(A) = 1$.

(ii) Suppose $h_y(\overline{A}) = 1$; then $h_y(\overline{P \wedge Q}) = 1$; so by HIx($\wedge$), $h_y(\overline{P}) = 1$ and $h_y(\overline{Q}) = 1$; so by assumption, $h_x(\overline{P}) = 1$ and $h_x(\overline{Q}) = 1$; so by HIx($\wedge$), $h_x(\overline{P \wedge Q}) = 1$; so $h_x(\overline{A}) = 1$.

($\vee$)

($\sqsupset$) Suppose A is $P \sqsupset Q$. (i) Suppose $h_x(A) = 1$ but $h_y(A) = 0$; then $h_x(P \sqsupset Q) = 1$ but $h_y(P \sqsupset Q) = 0$. From the former, by HIx($\sqsupset$), any w such that xRw has $h_w(P) = 0$ or $h_w(Q) = 1$. From the latter, by HIx($\sqsupset$), there is some $z \in W$ such that yRz where $h_z(P) = 1$ and $h_z(Q) = 0$. But xRy and yRz; so by τ, xRz; so $h_z(P) = 0$ or $h_z(Q) = 1$. This is impossible; reject the assumption: if $h_x(A) = 1$, then $h_y(A) = 1$.

(ii) Suppose $h_y(\overline{A}) = 1$; then $h_y(\overline{P \sqsupset Q}) = 1$; so by HIx($\sqsupset$), $h_y(P) = 0$ or $h_y(\overline{Q}) = 1$; so by assumption, $h_x(P) = 0$ or $h_x(\overline{Q}) = 1$; so by HIx($\sqsupset$), $h_x(\overline{P \sqsupset Q}) = 1$; so $h_x(\overline{A}) = 1$.

($\sqsupset$)$_W$ Suppose A is $P \sqsupset Q$. (i) Suppose $h_x(A) = 1$ but $h_y(A) = 0$; then $h_x(P \sqsupset Q) = 1$ but $h_y(P \sqsupset Q) = 0$. From the former, by HIx($\sqsupset$)$_W$, any w such that xRw has $h_w(P) = 0$ or $h_w(Q) = 1$. From the latter, by HIx($\sqsupset$)$_W$, there is some $z \in W$ such that yRz where $h_z(P) = 1$ and $h_z(Q) = 0$. But xRy and yRz; so by τ, xRz; so $h_z(P) = 0$ or $h_z(Q) = 1$. This is impossible; reject the assumption: if $h_x(A) = 1$, then $h_y(A) = 1$.

(ii) Suppose $h_y(\overline{A}) = 1$ but $h_x(\overline{A}) = 0$; then $h_y(\overline{P \sqsupset Q}) = 1$ but $h_x(\overline{P \sqsupset Q}) = 0$. From the former, by HIx($\sqsupset$)$_W$, there is some $w \in W$ such that yRw and $h_w(P) = 1$ and $h_w(\overline{Q}) = 0$. But xRy and yRw; so by τ, xRw; so there is some $w \in W$ such that xRw and $h_w(P) = 1$ and $h_w(\overline{Q}) = 0$; but since $h_x(\overline{P \sqsupset Q}) = 0$, by HIx($\sqsupset$)$_W$, any z such that xRz has $h_z(P) = 0$ or $h_z(\overline{Q}) = 1$; so since xRw, $h_w(P) = 0$ or $h_w(\overline{Q}) = 1$. This is impossible; reject the assumption: if $h_y(\overline{A}) = 1$, then $h_x(\overline{A}) = 1$.

For any such A, if $h_x(A) = 1$, then $h_y(A) = 1$, and if $h_y(\overline{A}) = 1$, then $h_x(\overline{A}) = 1$.

So, returning to the left-hand case for (H$_\text{I}$), $h_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Ix} A_t$, which is to say, $\Gamma_k \models^*_{Ix} \mathcal{P}_k$. And similarly in the right-hand case.

(D) If $\mathcal{P}_k$ arises by D, then the picture is like this,

$$\begin{array}{r|l} i & A_s \\ & \\ k & \overline{A}_s \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $\overline{A}_s$. Where this rule is included in *NIx*, *Ix* has condition *exc*, so no interpretation has $h_x(p) = \{1, 0\}$. By assumption, $\Gamma_i \models^*_{Ix} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L7.1*a*, $\Gamma_k \models^*_{Ix} A_s$. Suppose $\Gamma_k \not\models^*_{Ix} \overline{A}_s$; then by VIx*, there is an *Ix* interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\overline{A}) = 0$; since $h(\Gamma_k) = 1$, by VIx*, $h_{m(s)}(A) = 1$. But for these interpretations, for any A, if $h_x(A) = 1$ then $h_x(\overline{A}) = 1$.

Basis: A is a parameter p. Suppose $h_x(A) = 1$; then $h_x(p) = 1$; so $1 \in h_x(p)$; so by *exc*, $0 \notin h_x(p)$; so $h_x(\overline{p}) = 1$; so $h_x(\overline{A}) = 1$.

Assp: For any i, $0 \leq i < k$, if A has i operators, and $h_x(A) = 1$, then $h_x(\overline{A}) = 1$.

Show: If A has k operators, and $h_x(A) = 1$, then $h_x(\overline{A}) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \sqsupset Q$, where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. Suppose $h_x(A) = 1$; then $h_x(\neg P) = 1$; so by HIx($\neg$), $h_x(\overline{P}) = 0$; so by assumption, $h_x(P) = 0$; so by HIx($\neg$), $h_x(\overline{\neg P}) = 1$, which is to say, $h_x(\overline{A}) = 1$.

($\wedge$) A is $P \wedge Q$. Suppose $h_x(A) = 1$; then $h_x(P \wedge Q) = 1$; so by HIx($\wedge$), $h_x(P) = 1$ and $h_x(Q) = 1$; so by assumption, $h_x(\overline{P}) = 1$ and $h_x(\overline{Q}) = 1$; so by HIx($\wedge$), $h_x(\overline{P \wedge Q}) = 1$, which is to say $h_x(\overline{A}) = 1$.

($\vee$)

($\sqsupset$) A is $P \sqsupset Q$. Suppose $h_x(A) = 1$; then $h_x(P \sqsupset Q) = 1$; so by HIx($\sqsupset$), for any w such that xRw, $h_w(P) = 0$ or $h_w(Q) = 1$; but by ρ, xRx; so $h_x(P) = 0$ or $h_x(Q) = 1$; so by assumption, $h_x(P) = 0$ or $h_x(\overline{Q}) = 1$; so by HIx($\sqsupset$), $h_x(\overline{P \sqsupset Q}) = 1$, which is to say $h_x(\overline{A}) = 1$.

For any A, if $h_x(A) = 1$, then $h_x(\overline{A}) = 1$.

So, returning to the case for (D), $h_{m(s)}(\overline{A}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Ix} \overline{A}$, which is to say, $\Gamma_k \models^*_{Ix} \mathcal{P}_k$.

($\overline{\sqsupset\mathrm{I}}$) If $\mathcal{P}_k$ arises by $\overline{\sqsupset\mathrm{I}}$, then the picture is like this,

$$\begin{array}{r|l|l}
 & & A_s \\ \cline{3-3}
 & & \\
i & & \overline{B}_s \\
k & (\overline{A \sqsupset B})_s &
\end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $(\overline{A \sqsupset B})_s$. By assumption, $\Gamma_i \models^*_{Ix} \overline{B}_s$; and by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_s\}$; so by L7.1$a$, $\Gamma_k \cup \{A_s\} \models^*_{Ix} \overline{B}_s$. Suppose $\Gamma_k \not\models^*_{Ix} (\overline{A \sqsupset B})_s$; then by VIx*, there is some Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\overline{A \sqsupset B}) = 0$; from the latter, by HIx($\sqsupset$), $h_{m(s)}(A) = 1$ and $h_{m(s)}(\overline{B}) = 0$; so $h_m(\Gamma_k) = 1$ and $h_{m(s)}(A) = 1$; so $h_m(\Gamma_k \cup \{A_s\}) = 1$; so by VIx*, $h_{m(s)}(\overline{B}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Ix} (\overline{A \sqsupset B})_s$, which is to say, $\Gamma_k \models^*_{Ix} \mathcal{P}_k$.

($\overline{\sqsupset\mathrm{E}}$) If $\mathcal{P}_k$ arises by $\overline{\sqsupset\mathrm{E}}$, then the picture is like this,

$$\begin{array}{r|l}
i & (\overline{A \sqsupset B})_s \\
j & A_s \\
 & \\
k & \overline{B}_s
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\overline{B}_s$. By assumption, $\Gamma_i \models^*_{Ix} (\overline{A \sqsupset B})_s$ and $\Gamma_j \models^*_{Ix} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L7.1$a$, $\Gamma_k \models^*_{Ix} (\overline{A \sqsupset B})_s$ and $\Gamma_k \models^*_{Ix} A_s$. Suppose $\Gamma_k \not\models^*_{Ix} \overline{B}_s$; then by VIx*, there is some Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\overline{B}) = 0$; since $h_m(\Gamma_k) = 1$, by VIx*, $h_{m(s)}(\overline{A \sqsupset B}) = 1$ and $h_{m(s)}(A) = 1$; from the former, by HIx($\sqsupset$), $h_{m(s)}(A) = 0$ or $h_{m(s)}(\overline{B}) = 1$; so $h_{m(s)}(\overline{B}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Ix} \overline{B}_s$, which is to say, $\Gamma_k \models^*_{Ix} \mathcal{P}_k$.

($\overline{\sqsupset\mathrm{I}}_W$) If $\mathcal{P}_k$ arises by $\overline{\sqsupset\mathrm{I}}_W$, then the picture is like this,

$$\begin{array}{r|l}
h & s.t \\
i & A_t \\
j & \neg B_t \\
 & \\
k & (\overline{A \sqsupset B})_s
\end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is $(\overline{A \sqsupset B})_s$. By assumption, $\Gamma_h \models^*_{Ix} s.t$, $\Gamma_i \models^*_{Ix} A_t$ and $\Gamma_j \models^*_{Ix} \neg B_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L7.1$a$, $\Gamma_k \models^*_{Ix} s.t$, $\Gamma_k \models^*_{Ix} A_t$, and $\Gamma_k \models^*_{Ix} \neg B_t$. Suppose $\Gamma_k \not\models^*_{Ix} (\overline{A \sqsupset B})_s$; then by VIx*, there is some Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\overline{A \sqsupset B}) = 0$; since $h_m(\Gamma_k) = 1$, by VIx*, $\langle m(s), m(t)\rangle \in R$, $h_{m(t)}(A) = 1$ and $h_{m(t)}(\neg B) = 1$; from the last of these, by HIx($\neg$), $h_{m(t)}(\overline{B}) = 0$; so there is some $m(t) \in W$ such that $m(s)Rm(t)$ and $h_{m(t)}(A) = 1$ and $h_{m(t)}(\overline{B}) = 0$; so by HIx$(\sqsupset)_W$, $h_{m(s)}(\overline{A \sqsupset B}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Ix} (\overline{A \sqsupset B})_s$, which is to say, $\Gamma_k \models^*_{Ix} \mathcal{P}_k$.

$(\overline{\sqsupset\mathrm{E}}_W)$ If $\mathcal{P}_k$ arises by $\overline{\sqsupset\mathrm{E}}_W$, then the picture is like this,

$$\begin{array}{r|l}
i & (\overline{A \sqsupset B})_s \\
 & \;\vert\; s.t \\
 & \;\vert\; A_t \\
 & \;\vert\; \underline{\neg B_t} \\
 & \;\vert\; \\
 & \;\vert\; \\
j & \;\vert\; C_u \\
k & C_u
\end{array}$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is C_u. By assumption, $\Gamma_i \models^*_{Ix} (\overline{A \sqsupset B})_s$ and $\Gamma_j \models^*_{Ix} C_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s.t, A_t, \neg B_t\}$; so by L7.1$a$, $\Gamma_k \models^*_{Ix} (\overline{A \sqsupset B})_s$ and $\Gamma_k \cup \{s.t, A_t, \neg B_t\} \models^*_{Ix} C_u$. Suppose $\Gamma_k \not\models^*_{Ix} C_u$; then by VIx*, there is an Ix interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(u)}(C) = 0$; since $h_m(\Gamma_k) = 1$, by VIx*, $h_{m(s)}(\overline{A \sqsupset B}) = 1$; so, by HIx$(\sqsupset)_W$, there is some $w \in W$ such that $m(s)Rw$ and $h_w(A) = 1$ and $h_w(\overline{B}) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, h\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$, $m(s)Rm'(t)$ and $h_{m'(t)}(A) = 1$ and $h_{m'(t)}(\overline{B}) = 0$; from the last of these, by HIx($\neg$), $h_{m'(t)}(\neg B) = 1$; so $h_{m'}(\Gamma_k \cup \{s.t, A_t, \neg B\}) = 1$; so by VIx*, $h_{m'(u)}(C) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $h_{m(u)}(C) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Ix} C_u$, which is to say, $\Gamma_k \models^*_{Ix} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{Ix} \mathcal{P}_i$.

THEOREM 7.4 *NIx is complete: if* $\Gamma \models_{Ix} A$ *then* $\Gamma \vdash_{NIx} A$.

Suppose $\Gamma \models_{Ix} A$; then $\Gamma_0 \models^*_{Ix} A_0$; we show that $\Gamma_0 \vdash^*_{NIx} A_0$. As usual, this reduces to the standard notion. For the following, fix on some particular Ix. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NIx} /A/_s$ and $\Gamma \vdash^*_{NIx} \backslash\neg A\backslash_s$.

L7.2*a* If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NIx} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

Reasoning as in L7.2.

L7.3*a* There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$ with access relations $s.t$.

Proof by construction as usual.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NIx} /A/_s$ or $\Gamma \vdash^*_{NIx} \backslash\neg A\backslash_s$.

SGT Γ is a SCAPEGOAT set for $(\sqsupset)_{I_{3,4}}$ iff for every formula of the form $\overline{\neg(A \sqsupset B)}_s$, if $\Gamma \vdash^*_{NIx} \overline{\neg(A \sqsupset B)}_s$ then there is some t such that $\Gamma \vdash^*_{NIx} s.t$, $\Gamma \vdash^*_{NIx} A_t$ and $\Gamma \vdash^*_{NIx} \overline{\neg B}_t$.

Γ is a SCAPEGOAT set for $(\sqsupset)_{I_W}$ iff for every formula of the form $\overline{(A \sqsupset B)}_s$, if $\Gamma \vdash^*_{NIx} \overline{\neg(A \sqsupset B)}_s$ then there is some t such that $\Gamma \vdash^*_{NIx} s.t$, $\Gamma \vdash^*_{NIx} A_t$ and $\Gamma \vdash^*_{NIx} \overline{\neg B}_t$; and if $\Gamma \vdash^*_{NIx} \overline{(A \sqsupset B)}_s$ then there is some t such that $\Gamma \vdash^*_{NIx} s.t$, $\Gamma \vdash^*_{NIx} A_t$ and $\Gamma \vdash^*_{NIx} \neg B_t$

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L7.3*a*, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the formulas, together with all the access relations $s.t$; let $\mathcal{E}_0$ be this enumeration. Then for the first expression $\mathcal{P}$ in $\mathcal{E}_{i-1}$ such that all its subscripts are 0 or introduced in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $\mathcal{P}$, and set,

$\Omega_i = \Omega_{i-1}$ if $\Omega_{i-1} \vdash^*_{NIx} \backslash\neg A\backslash_s$
$\Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\}$ if $\Omega_{i-1} \nvdash^*_{NIx} \backslash\neg A\backslash_s$

and

$I_{3,4}$: $\Omega_i = \Omega_{i^*}$ if A_s is not of the form $\overline{\neg(P \sqsupset Q)}_s$
$\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \overline{\neg Q}_t\}$ if A_s is of the form $\overline{\neg(P \sqsupset Q)}_s$
I_W: $\Omega_i = \Omega_{i^*}$ if A_s is not of the form $\overline{\neg(P \sqsupset Q)}_s$ or $(\overline{P \sqsupset Q})_s$
$\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \overline{\neg Q}_t\}$ if A_s is of the form $\overline{\neg(P \sqsupset Q)}_s$
$\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \neg Q_t\}$ if A_s is of the form $(\overline{P \sqsupset Q})_s$
-where t is the first subscript not included in Ω_{i^*}

then

$\Gamma' = \bigcup_{i \geq 0} \Omega_i$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s appears in Γ'; then there is some Ω_i in which it is first appears; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L7.4*a* For any s included in Γ', Γ' is s-maximal.

Reasoning as in L7.4.

L7.5*a* If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$, (iii) $\Omega_{k^*} \cup \{s.t, P_t, \overline{\neg Q}_t\}$ in $I_{3,4}$ or I_W, or (iv) $\Omega_{k^*} \cup \{s.t, P_t, \neg Q_t\}$ in I_W.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NIx} \backslash\neg A\backslash_s$; so by L7.2*a*, $\Omega_{k-1} \cup \{/A/_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, P_t, \overline{\neg Q}_t\}$ in $I_{3,4}$ or I_W. In this case, as above, Ω_{k*} is consistent and by construction, $\overline{\neg(P \sqsupset Q)}_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{s.t, P_t, \overline{\neg Q}_t\} \vdash^*_{NIx} /A/_u$ and $\Omega_{k^*} \cup \{s.t, P_t, \overline{\neg Q}_t\} \vdash^*_{NIx} \backslash\neg A\backslash_u$. So reason as follows,

1	$\vert\ \Omega_{k^*}$	
2	$\vert\ \vert\ s.t$	A (g, $\sqsupset$I)
3	$\vert\ \vert\ \underline{P_t}$	A (g, $\sqsupset$I)
4	$\vert\ \vert\ \vert\ \underline{\overline{\neg Q}_t}$	A (c, $\neg$E)
5	$\vert\ \vert\ \vert\ /A/_u$	from $\Omega_{k^*} \cup \{s.t, P_t, \overline{\neg Q}_t\}$
6	$\vert\ \vert\ \vert\ \backslash\neg A\backslash_u$	from $\Omega_{k^*} \cup \{s.t, P_t, \overline{\neg Q}_t\}$
7	$\vert\ \vert\ Q_t$	4-6 $\neg$E
8	$\vert\ (P \sqsupset Q)_s$	2-7 $\sqsupset$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NIx} (P \sqsupset Q)_s$; but $\overline{\neg(P \sqsupset Q)}_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NIx} \overline{\neg(P \sqsupset Q)}_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, P_t, \neg Q_t\}$ in I_W. In this case, as above, Ω_{k*} is consistent and by construction, $(\overline{P \sqsupset Q})_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{s.t, P_t, \neg Q_t\} \vdash^*_{NIx} /A/_u$ and $\Omega_{k^*} \cup \{s.t, P_t, \neg Q_t\} \vdash^*_{NIx} \backslash\neg A\backslash_u$. So reason as follows,

1	$\vert\ \Omega_{k^*}$	
2	$\vert\ \vert\ \underline{(\overline{P \sqsupset Q})_s}$	A (c, $\neg$I)
3	$\vert\ \vert\ \vert\ s.t$	A (g, $\overline{\sqsupset\mathrm{E}}_W$)
4	$\vert\ \vert\ \vert\ P_t$	A (g, $\overline{\sqsupset\mathrm{E}}_W$)
5	$\vert\ \vert\ \vert\ \underline{\neg Q_t}$	A (g, $\overline{\sqsupset\mathrm{E}}_W$)
6	$\vert\ \vert\ \vert\ /A/_u$	from $\Omega_{k^*} \cup \{s.t, P_t, \neg Q_t\}$
7	$\vert\ \vert\ /A/_u$	2,3-6 $\overline{\sqsupset\mathrm{E}}_W$
8	$\vert\ \vert\ \vert\ s.t$	A (g, $\overline{\sqsupset\mathrm{E}}_W$)
9	$\vert\ \vert\ \vert\ P_t$	A (g, $\overline{\sqsupset\mathrm{E}}_W$)
10	$\vert\ \vert\ \vert\ \underline{\neg Q_t}$	A (g, $\overline{\sqsupset\mathrm{E}}_W$)
11	$\vert\ \vert\ \vert\ \backslash\neg A\backslash_u$	from $\Omega_{k^*} \cup \{s.t, P_t, \neg Q_t\}$
12	$\vert\ \vert\ \backslash\neg A\backslash_u$	2,8-11 $\overline{\sqsupset\mathrm{E}}_W$
13	$\vert\ \neg(P \sqsupset Q)_s$	2-12 $\neg$I

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NIx} \neg(P \sqsupset Q)_s$; but $(\overline{P \sqsupset Q})_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L7.6a If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 and L6.6.

L7.7a If Γ_0 is consistent, then Γ' is a scapegoat set for $(\sqsupset)_{I_{3,4}}$ and $(\sqsupset)_{I_W}$.

For $(\sqsupset)_{I_{3,4}}$ and $(\sqsupset)_{I_W}$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NIx} \overline{\neg(P \sqsupset Q)}_s$. By L7.6$a$, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NIx} \neg\neg(P \sqsupset Q)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\overline{\neg(P \sqsupset Q)}_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \overline{\neg Q}_t\}$; so by construction, $s.t \in \Gamma'$, $P_t \in \Gamma'$ and $\overline{\neg Q}_t \in \Gamma'$; so $\Gamma' \vdash^*_{NIx} s.t$, $\Gamma' \vdash^*_{NIx} P_t$ and $\Gamma' \vdash^*_{NIx} \overline{\neg Q}_t$. So Γ' is a scapegoat set for $(\sqsupset)_{I_{3,4}}$.

Furthermore for $(\sqsupset)_{I_W}$. Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$. By L7.6$a$, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NIx} \neg(P \sqsupset Q)_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{(\overline{P \sqsupset Q})_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, P_t, \neg Q_t\}$; so by construction, $s.t \in \Gamma'$, $P_t \in \Gamma'$ and $\neg Q_t \in \Gamma'$; so $\Gamma' \vdash^*_{NIx} s.t$, $\Gamma' \vdash^*_{NIx} P_t$ and $\Gamma' \vdash^*_{NIx} \neg Q_t$. So Γ' is a scapegoat set for $(\sqsupset)_{I_W}$.

C(I) We construct an interpretation $I = \langle W, R, h \rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $\langle w_s, w_t \rangle \in R$ iff $\Gamma' \vdash^*_{NIx} s.t$ and $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{NIx} /p/_s$.

L7.8a If Γ_0 is consistent then for $\langle W, R, h \rangle$ constructed as above, and for any s included in Γ', $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NIx} /A/_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L7.4a, Γ' is s-maximal. By L7.6a and L7.7a, Γ' is consistent and a scapegoat set for the different conditionals. Now by induction on the number of operators in $/A/_s$,

Basis: If $/A/_s$ has no operators, then it is a parameter $/p/_s$ and by construction, $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{NIx} /p/_s$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NIx} /A/_s$.

Assp: For any i, $0 \leq i < k$, if $/A/_s$ has i operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NIx} /A/_s$.

Show: If $/A/_s$ has k operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NIx} /A/_s$.

If $/A/_s$ has k operators, then it is of the form $/\neg P/_s$, $/P \wedge Q/_s$, $/P \vee Q/_s$ or $/P \sqsupset Q/_s$, where P and Q have $< k$ operators.

($\neg$)

($\wedge$)

($\vee$)

($\sqsupset$) $/A/_s$ is $/P \sqsupset Q/_s$. (i) Suppose $h_{w_s}(/A/) = 1$ but $\Gamma' \nvdash^*_{NIx} /A/_s$; then $h_{w_s}(/P \sqsupset Q/) = 1$, but $\Gamma' \nvdash^*_{NIx} /P \sqsupset Q/_s$.

$I_{3,4}$: (a) $/P \sqsupset Q/_s$ is $(P \sqsupset Q)_s$. Then $h_{w_s}(P \sqsupset Q) = 1$, but $\Gamma' \nvdash^*_{NIx} (P \sqsupset Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NIx} \overline{\neg(P \sqsupset Q)}_s$; but since Γ' is a scapegoat set for $(\sqsupset)_{I_{3,4}}$, there is some t such that $\Gamma' \vdash^*_{NIx} s.t$, $\Gamma' \vdash^*_{NIx} P_t$ and $\Gamma' \vdash^*_{NIx} \overline{\neg Q}_t$; from the last of these, by consistency, $\Gamma' \nvdash^*_{NIx} Q_t$; so by assumption, $h_{w_t}(P) = 1$ and $h_{w_t}(Q) = 0$; and since $\Gamma' \vdash^*_{NIx} s.t$, by construction, $\langle w_s, w_t \rangle \in R$; so there is some $y \in W$ such that $w_s R y$ and $h_y(P) = 1$ but $h_y(Q) = 0$; so by HIx($\sqsupset$), $h_{w_s}(P \sqsupset Q) = 0$. This is impossible.

(b) $/P \sqsupset Q/_s$ is $(\overline{P \sqsupset Q})_s$. Then $h_{w_s}(\overline{P \sqsupset Q}) = 1$, but $\Gamma' \nvdash^*_{NIx} (\overline{P \sqsupset Q})_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NIx} \neg(P \sqsupset Q)_s$. From the former, by HIx($\sqsupset$), $h_{w_s}(P) = 0$ or $h_{w_s}(\overline{Q}) = 1$. Suppose the first; so $h_{w_s}(P) = 0$; then by assumption, $\Gamma' \nvdash^*_{NIx} P_s$; so by s-maximality, $\Gamma' \vdash^*_{NIx} \overline{\neg P}_s$; so by reasoning as follows,

$$\begin{array}{r|l|l|ll}
1 & \overline{\neg P}_s & & & \text{from } \Gamma' \\
2 & & P_s & & \text{A } (g, \overline{\sqsupset}\text{I}) \\
3 & & & \neg Q_s & \text{A } (c, \neg\text{E}) \\
4 & & & P_s & 2 \text{ R} \\
5 & & & \overline{\neg P}_s & 1 \text{ R} \\
6 & & \overline{Q}_s & & 3\text{-}5\ \neg\text{E} \\
7 & (\overline{P \sqsupset Q})_s & & & 2\text{-}6\ \overline{\sqsupset}\text{I}
\end{array}$$

$\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$; so $\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$ and $\Gamma' \vdash^*_{NIx} \neg(P \sqsupset Q)_s$; so Γ' is inconsistent. Suppose the second; so $h_{w_s}(\overline{Q}) = 1$; then by assumption, $\Gamma' \vdash^*_{NIx} \overline{Q}_s$; so by reasoning as follows,

$$\begin{array}{r|l|ll}
1 & \overline{Q}_s & & \text{from } \Gamma' \\
2 & & P_s & \text{A } (g, \overline{\sqsupset}\text{I}) \\
3 & & \overline{Q}_s & 1 \text{ R} \\
4 & (\overline{P \sqsupset Q})_s & & 2\text{-}3\ \overline{\sqsupset}\text{I}
\end{array}$$

$\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$; so $\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$ and $\Gamma' \vdash^*_{NIx} \neg(P \sqsupset Q)_s$; so Γ' is inconsistent. In either case, Γ' is inconsistent. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^*_{NIx} /A/_s$.

I_W: (a) $/P \sqsupset Q/_s$ is $(P \sqsupset Q)_s$. Same reasoning as in $I_{3,4}$.
(b) $/P \sqsupset Q/_s$ is $(\overline{P \sqsupset Q})_s$. Then $h_{w_s}(\overline{P \sqsupset Q}) = 1$, but $\Gamma' \nvdash^*_{NIx} (\overline{P \sqsupset Q})_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NIx} \neg(P \sqsupset Q)_s$. From the former, by HIx($\sqsupset$)$_W$, there is some $w_t \in W$ such that $w_s R w_t$ and $h_{w_t}(P) = 1$ and $h_{w_t}(\overline{Q}) = 0$; so by construction and by assumption, $\Gamma' \vdash^*_{NIx} s.t$ and $\Gamma' \vdash^*_{NIx} P_t$ but $\Gamma' \nvdash^*_{NIx} \overline{Q}_t$; from the last of these, by s-maximality, $\Gamma' \vdash^*_{NIx} \neg Q_t$; so by reasoning as follows,

$$\begin{array}{r|ll}
1 & s.t & \text{from } \Gamma' \\
2 & P_t & \text{from } \Gamma' \\
3 & \neg Q_t & \text{from } \Gamma' \\
4 & (\overline{P \sqsupset Q})_s & 1,2,3\ \overline{\sqsupset}\text{I}_W
\end{array}$$

$\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$; so $\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$ and $\Gamma' \vdash^*_{NIx} \neg(P \sqsupset Q)_s$; so Γ' is inconsistent. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^*_{NIx} /A/_s$.

So in both these cases, if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^*_{NIx} /A/_s$.

(ii) Suppose $\Gamma' \vdash^*_{NIx} /A/_s$ but $h_{w_s}(/A/) = 0$; then $\Gamma' \vdash^*_{NIx} /P \sqsupset Q/_s$ but $h_{w_s}(/P \sqsupset Q/) = 0$.

$I_{3,4}$: (a) $/P \sqsupset Q/_s$ is $(P \sqsupset Q)_s$. Then $\Gamma' \vdash^*_{NIx} (P \sqsupset Q)_s$ but $h_{w_s}(P \sqsupset Q) = 0$. From the latter, by HIx($\sqsupset$), there is some $w_t \in W$ such that $w_s R w_t$ and $h_{w_t}(P) = 1$ but $h_{w_t}(Q) = 0$; so by assumption, $\Gamma' \vdash^*_{NIx} s.t$ and $\Gamma' \vdash^*_{NIx} P_t$ but $\Gamma' \nvdash^*_{NIx} Q_t$; from the last of these, by s-maximality, $\Gamma' \vdash^*_{NIx} \overline{\neg Q}_t$. So by reasoning as follows,

$$\begin{array}{r|l|ll}
1 & \Gamma' & & \text{from } \Gamma' \\
2 & & \underline{(P \sqsupset Q)_s} & \text{A } (c, \neg\text{I}) \\
3 & & s.t & \text{from } \Gamma' \\
4 & & P_t & \text{from } \Gamma' \\
5 & & Q_t & 2,3,4\ \sqsupset\text{E} \\
6 & & \overline{\neg Q}_t & \text{from } \Gamma' \\
7 & \overline{\neg(P \sqsupset Q)}_s & & 2\text{-}6\ \neg\text{I}
\end{array}$$

$\Gamma' \vdash^*_{NIx} \overline{\neg(P \sqsupset Q)}_s$; so by consistency, $\Gamma' \nvdash^*_{NIx} (P \sqsupset Q)_s$. This is impossible.
(b) $/P \sqsupset Q/_s$ is $(\overline{P \sqsupset Q})_s$. Then $\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$ but $h_{w_s}(\overline{P \sqsupset Q}) = 0$. From the latter, by HIx($\sqsupset$),

$h_{w_s}(P) = 1$ but $h_{w_s}(\overline{Q}) = 0$; so by assumption, $\Gamma' \vdash^*_{NIx} P_s$ but $\Gamma' \nvdash^*_{NIx} \overline{Q}_s$; so by s-maximality, $\Gamma' \vdash^*_{NIx} \neg Q_s$. So by reasoning as follows,

$$\begin{array}{r|lll}
1 & \Gamma' & & \text{from } \Gamma' \\
2 & & \underline{(\overline{P \sqsupset Q})_s} & \text{A } (c, \neg\text{I}) \\
3 & & P_s & \text{from } \Gamma' \\
4 & & \overline{Q}_s & 2,3\ \overline{\sqsupset\text{E}} \\
5 & & \neg Q_s & \text{from } \Gamma' \\
6 & \neg(P \sqsupset Q)_s & & \text{2-5 } \neg\text{I}
\end{array}$$

$\Gamma' \vdash^*_{NIx} \neg(P \sqsupset Q)_s$; so by consistency, $\Gamma' \nvdash^*_{NIx} (\overline{P \sqsupset Q})_s$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NIx} /A/_s$, then $h_{w_s}(/A/) = 1$.

I_W: (a) $/P \sqsupset Q/_s$ is $(P \sqsupset Q)_s$. Same reasoning as in $I_{3,4}$.
(b) $/P \sqsupset Q/_s$ is $(\overline{P \sqsupset Q})_s$. Then $\Gamma' \vdash^*_{NIx} (\overline{P \sqsupset Q})_s$ but $h_{w_s}(\overline{P \sqsupset Q}) = 0$. From the former, since Γ' is a scapegoat set for $(\sqsupset)_{I_W}$, there is some t such that $\Gamma' \vdash^*_{NIx} s.t$, $\Gamma' \vdash^*_{NIx} P_t$ and $\Gamma' \vdash^*_{NIx} \neg Q_t$. Since $h_{w_s}(\overline{P \sqsupset Q}) = 0$, by HIx$(\sqsupset)_W$, for any $y \in W$ such that $w_s R y$, $h_y(P) = 0$ or $h_y(\overline{Q}) = 1$; so if $w_s R w_t$ then either $h_{w_t}(P) = 0$ or $h_{w_t}(\overline{Q}) = 1$; so by construction and assumption, if $\Gamma' \vdash^*_{NIx} s.t$ then either $\Gamma' \nvdash^*_{NIx} P_t$ or $\Gamma' \vdash^*_{NIx} \overline{Q}_t$; so, since $\Gamma' \vdash^*_{NIx} s.t$, either $\Gamma' \nvdash^*_{NIx} P_t$ or $\Gamma' \vdash^*_{NIx} \overline{Q}_t$; so, since $\Gamma' \vdash^*_{NIx} P_t$, $\Gamma' \vdash^*_{NIx} \overline{Q}_t$. But $\Gamma' \vdash^*_{NIx} \neg Q_t$; so, by consistency $\Gamma' \nvdash^*_{NIx} \overline{Q}_t$. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NIx} /A/_s$, then $h_{w_s}(/A/) = 1$.

So in both these cases, if $\Gamma' \vdash^*_{NIx} /A/_s$ then $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NIx} /A/_s$.

For any A_s, $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NIx} /A/_s$.

L7.9*a* If Γ_0 is consistent, then $\langle W, R, h \rangle$ constructed as above is an Ix interpretation.

For this, we need to show that the interpretation meets the ρ, τ and h conditions.

(ρ) Suppose $w_s \in W$. Then by construction, s is a subscript in Γ'; so by (AMρ), $\Gamma' \vdash^*_{NIx} s.s$; so by construction, $\langle w_s, w_s \rangle \in R$ and ρ is satisfied.

(τ)

(h) Suppose $w_s R w_t$, $v_{w_s}(p) = 1$, and $v_{w_t}(\overline{p}) = 1$. Then by construction, $\Gamma' \vdash^*_{NIx}$ $s.t$, $\Gamma' \vdash^*_{NIx} p_s$, and $\Gamma' \vdash^*_{NIx} \overline{p}_t$; so by (H), $\Gamma' \vdash^*_{NIx} p_t$ and $\Gamma' \vdash^*_{NIx} \overline{p}_s$; so by construction, $v_{w_t}(p) = 1$ and $v_{w_s}(\overline{p}) = 1$ and h is satisfied.

Map For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L7.10*a* If Γ_0 is consistent, then $h_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 and L6.9.

Main result: Suppose $\Gamma \models_{Ix} A$ but $\Gamma \nvdash_{NIx} A$. Then $\Gamma_0 \models^*_{Ix} A_0$ but $\Gamma_0 \nvdash^*_{NIx} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NIx} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NIx} A_0$; so $\Gamma_0 \nvdash_{NIx} *]\neg\neg A_0$; so by L7.2*a*, $\Gamma_0 \cup \{\overline{\neg A_0}\}$ is consistent; so by L7.9*a* and L7.10*a*, there is an *Ix* interpretation $\langle W, R, h\rangle_m$ constructed as above such that $h_m(\Gamma_0 \cup \{\overline{\neg A_0}\}) = 1$; so $h_{m(0)}(\overline{\neg A}) = 1$; so by H$\upsilon(\neg)$, $h_{m(0)}(A) = 0$; so $h_m(\Gamma_0) = 1$ and $h_{m(0)}(A) = 0$; so by VIx*, $\Gamma_0 \not\models^*_{Ix} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{Ix} A$, then $\Gamma \vdash_{NIx} A$.

8 Mainstream Relevant Logics: *Bx* (ch. 10,11)

The treatment here for Priest's chapter 11 is minimal: there are only resources for *CK* with applications in chapter 11, as well as chapter 10. I follow Priest in developing the star-semantics on its own terms, and pick up the four-valued semantics again in the next section.

8.1 Language / Semantic Notions

LBx The vocabulary consists of propositional parameters $p_0, p_1 \ldots$ with the operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, (and $>$). Each propositional parameter is a formula; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A > B)$. $A \supset B$ abbreviates $\neg A \vee B$, and $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

IBrx Without '$>$' in the language, an interpretation is $\langle W, N, R, *, \preceq, v\rangle$ where W is a set of worlds; N is a subset of W; R is a subset of $W^3 = W \times W \times W$; $*$ is a function from worlds to worlds such that $w^{**} = w$; and v is a function such that for any $w \in W$ and p,

$v_w(p) = 1$ or $v_w(p) = 0$. $\preceq$ is a reflexive and transitive relation on W such that if $a \preceq b$ then $a \trianglelefteq b$ and $b^* \trianglelefteq a^*$, where,

$$a \trianglelefteq b = \begin{cases} \text{if } v_a(p) = 1 \text{ then } v_b(p) = 1 \\ \text{if } bRxy \text{ and } a \notin N, \text{ then } aRxy \\ \text{if } bRxy \text{ and } a \in N \text{ then } x \preceq y \end{cases}$$

As a constraint on interpretations, we require also,

NC For any $w \in N$, $wRxy$ iff $x = y$

Where x is empty or indicates some combination of the following constraints,

(C8) If $Rabc$, then Rac^*b^*

(C9) If there is an x such that $Rabx$ and $Rxcd$ then there is a y such that $Racy$ and $Rbyd$

(C10) If there is an x such that $Rabx$ and $Rxcd$ then there is a y such that $Rbcy$ and $Rayd$

(C11) If $Rabc$ then there is an x such that $Rabx$ and $Rxbc$

(C12) If $Rabc$ then there is an x such that $a \preceq x$ and $Rbxc$

(C13) If $x \in N$, $x^* \preceq x$.

(C14) For any x, if $x \in N$, $x^* \preceq x$, and if $x \notin N$, xRx^*x.

(C15) If $Rabc$ then $a \preceq c$.

(C16) If $Rabc$ then $a \preceq c$ or $b \preceq c$.

$\langle W, N, R, *, \preceq, v\rangle$ is a Bx interpretation when it meets the constraints from x. System B has none of the extra constraints; other systems add from the extra constraints as described in Priest. In particular, B_R is B_{C8-C12}.

IBcx When '>' is in the language, an interpretation is $\langle W, N, R, \{R_A \mid A \in \Im\}, *, v\rangle$, where $\Im$ is the set of all formulas and R_A is a subset of W^2. Condition NC remains in place, but none of C8 - C16. That is all for B_C (what Priest calls C_B). Where $f_A(w) = \{x \in W \mid wR_Ax\}$, and $[A] = \{x \in W \mid v_x(A) = 1\}$, B_{C^+} adds the constraints,

(1) For any $w \in N$, $f_A(w) \subseteq [A]$

(2) For any $w \in N$, if $w \in [A]$, then $w \in f_A(w)$

TB For complex expressions,

($\neg$) $v_w(\neg A) = 1$ if $v_{w^*}(A) = 0$, and 0 otherwise.

($\wedge$) $v_w(A \wedge B) = 1$ if $v_w(A) = 1$ and $v_w(B) = 1$, and 0 otherwise.

($\vee$) $v_w(A \vee B) = 1$ if $v_w(A) = 1$ or $v_w(B) = 1$, and 0 otherwise.

($\rightarrow$) $v_w(A \rightarrow B) = 1$ iff there are no $x, y \in W$ such that $wRxy$ and $v_x(A) = 1$ but $v_y(B) = 0$.

($>$) $v_w(A > B) = 1$ iff there is no $x \in W$ such that wR_Ax and $v_x(B) = 0$.

For a set Γ of formulas, $v_w(\Gamma) = 1$ iff $v_w(A) = 1$ for each $A \in \Gamma$; then,

VBx $\Gamma \models_{Bx} A$ iff there is no Bx interpretation $\langle W, N, R, *, \sqsubseteq, v\rangle$ / $\langle W, N, R, \{R_A \mid A \in \Im\}, *, v\rangle$ and $w \in N$ such that $v_w(\Gamma) = 1$ and $v_w(A) = 0$.

8.2 Natural Derivations: *NBx*

Allow subscripts of the sort i and $i^{\#}$. Where s is a subscript i or $i^{\#}$, $\overline{s}$ is the other. Say s is "introduced" as a subscript when either s or $\overline{s}$ is a subscript. For subscripts s, t, u allow also expressions of the sort $s \simeq t$, $s.t.u$ and $A_{s/t}$. Let $\mathcal{P}(s)$ be any expression in which s appears, and $\mathcal{P}(t)$ the same expression with one or more instances of s replaced by t.

$$\mathbf{R}\ \begin{array}{|l} P_s \\ \\ P_s \end{array} \qquad \neg\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{P_{\overline{s}}} \\ \\ Q_t \\ \neg Q_{\overline{t}} \end{array} \\ \neg P_s \end{array} \qquad \neg\mathbf{E}\ \begin{array}{|l} \begin{array}{|l} \underline{\neg P_{\overline{s}}} \\ \\ Q_t \\ \neg Q_{\overline{t}} \end{array} \\ P_s \end{array}$$

$$\wedge\mathbf{I}\ \begin{array}{|l} P_s \\ Q_s \\ \\ (P \wedge Q)_s \end{array} \qquad \wedge\mathbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ P_s \end{array} \qquad \wedge\mathbf{E}\ \begin{array}{|l} (P \wedge Q)_s \\ \\ Q_s \end{array}$$

$$\vee\mathbf{I}\ \begin{array}{|l} P_s \\ \\ (P \vee Q)_s \end{array} \qquad \vee\mathbf{I}\ \begin{array}{|l} P_s \\ \\ (Q \vee P)_s \end{array} \qquad \vee\mathbf{E}\ \begin{array}{|l} (P \vee Q)_s \\ \begin{array}{|l} \underline{P_s} \\ \\ R_t \end{array} \\ \begin{array}{|l} \underline{Q_s} \\ \\ R_t \end{array} \\ R_t \end{array}$$

$$\supset\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{P_{\overline{s}}} \\ \\ Q_s \end{array} \\ (P \supset Q)_s \end{array} \qquad \supset\mathbf{E}\ \begin{array}{|l} (P \supset Q)_s \\ P_{\overline{s}} \\ \\ Q_s \end{array}$$

$$\equiv\mathbf{I}\;\begin{array}{|l} \begin{array}{|l} P_{\overline{s}} \\ \hline \\ Q_s \end{array} \\ \begin{array}{|l} Q_{\overline{s}} \\ \hline \\ P_s \end{array} \\ (P \equiv Q)_s \end{array} \qquad \equiv\mathbf{E}\;\begin{array}{|l} (P \equiv Q)_s \\ P_{\overline{s}} \\ \\ Q_s \end{array} \qquad \equiv\mathbf{E}\;\begin{array}{|l} (P \equiv Q)_s \\ Q_{\overline{s}} \\ \\ P_s \end{array}$$

$$\rightarrow\mathbf{I}\;\begin{array}{|l} \begin{array}{|l} s.t.u \\ P_t \\ \hline \\ Q_u \end{array} \\ (P \rightarrow Q)_s \end{array} \qquad \rightarrow\mathbf{E}\;\begin{array}{|l} s.t.u \\ (P \rightarrow Q)_s \\ P_t \\ \\ Q_u \end{array} \qquad \not\rightarrow\mathbf{I}\;\begin{array}{|l} \overline{s}.t.u \\ P_t \\ \neg Q_{\overline{u}} \\ \\ \neg(P \rightarrow Q)_s \end{array} \qquad \not\rightarrow\mathbf{E}\;\begin{array}{|l} \neg(P \rightarrow Q)_s \\ \begin{array}{|l} \overline{s}.t.u \\ P_t \\ \neg Q_{\overline{u}} \\ \hline \\ R_v \end{array} \\ R_v \end{array}$$

where t and u are not introduced in any undischarged premise or assumption

where t and u are not introduced in any undischarged premise or assumption or by v

$$0\mathbf{I}\;\begin{array}{|l} s \simeq t \\ \\ 0.s.t \end{array} \qquad 0\mathbf{E}\;\begin{array}{|l} 0.s.t \\ \\ s \simeq t \end{array} \qquad \simeq\mathbf{I}\;\begin{array}{|l} \\ \\ s \simeq s \end{array} \qquad \simeq\mathbf{E}\;\begin{array}{|l} s \simeq t \\ \mathcal{P}(s) \\ \\ \mathcal{P}(t) \end{array} \quad \begin{array}{|l} s \simeq t \\ \mathcal{P}(\overline{s}) \\ \\ \mathcal{P}(\overline{t}) \end{array}$$

These are the rules of *NB*, where $\supset$I, $\supset$E, $\equiv$I, $\equiv$E and, as we shall see, $\not\rightarrow$I and $\not\rightarrow$E are derived. With $s \simeq t$, we can introduce $s \simeq s$ by $\simeq$I, and then get $t \simeq s$ by $\simeq$E; so informally, we let $\simeq$E include also a derived rule that reverses order around '$\simeq$' – using $s \simeq t$ to replace some instance(s) of t ($\overline{t}$) with s ($\overline{s}$). As usual, subscripts are 0 or introduced in an assumption that requires new subscripts (and similarly for the following). To make things easier to follow, cite lines for $\rightarrow$E only in the order listed above: first access, then the conditional, then the antecedent.

For relevant systems NB_x, allow expressions of the sort, $s \preceq t$ and $s \not\simeq t$. The latter contradicts $s \simeq t$ in $\neg$I and $\neg$E.[7] Then include rules from the following as appropriate.

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[7] We might allow a generic subscript z such that any $s \simeq t$ is $(s \simeq t)_z$ and $s \not\simeq t$ is $\neg(s \simeq t)_{z\#}$. Then the negation rules apply as stated.

$$\textbf{AM9}\;\begin{array}{|l} s.t.x \\ x.u.v \\ \begin{array}{|l} s.u.y \\ \underline{t.y.v} \\ \\ P_w \end{array} \\ P_w \end{array} \qquad \textbf{AM10}\;\begin{array}{|l} s.t.x \\ x.u.v \\ \begin{array}{|l} t.u.y \\ \underline{s.y.v} \\ \\ P_w \end{array} \\ P_w \end{array} \qquad \textbf{AM11}\;\begin{array}{|l} s.t.u \\ \begin{array}{|l} s.t.y \\ \underline{y.t.u} \\ \\ P_w \end{array} \\ P_w \end{array} \qquad \textbf{AM12}\;\begin{array}{|l} s.t.u \\ \begin{array}{|l} s \preceq y \\ \underline{t.y.u} \\ \\ P_w \end{array} \\ P_w \end{array}$$

$$\textbf{AM13}\;\begin{array}{|l} \\ \\ 0^{\#} \preceq 0 \end{array} \qquad \textbf{AM14}\;\begin{array}{|l} \begin{array}{|l} s \simeq 0 \\ \underline{\overline{s} \preceq s} \\ \\ P_w \end{array} \\ \begin{array}{|l} s \not\simeq 0 \\ \underline{s.\overline{s}.s} \\ \\ P_w \end{array} \\ P_w \end{array} \qquad \textbf{AM15}\;\begin{array}{|l} s.t.u \\ \\ s \preceq u \end{array} \qquad \textbf{AM16}\;\begin{array}{|l} s.t.u \\ \begin{array}{|l} \underline{s \preceq u} \\ \\ P_w \end{array} \\ \begin{array}{|l} \underline{t \preceq u} \\ \\ P_w \end{array} \\ P_w \end{array}$$

$$\textbf{AM8}\;\begin{array}{|l} s.t.u \\ \\ s.\overline{u}.\overline{t} \end{array} \qquad \preceq\!\textbf{E}\;\begin{array}{|l} s \preceq t \\ P_s \\ \\ P_t \end{array} \qquad \preceq^{\#}\;\begin{array}{|l} s \preceq t \\ \\ \overline{t} \preceq \overline{s} \end{array} \qquad \preceq^{R}\;\begin{array}{|l} s \not\simeq 0 \\ s \preceq t \\ t.u.v \\ \\ s.u.v \end{array} \quad \begin{array}{|l} s \simeq 0 \\ s \preceq t \\ t.u.v \\ \\ u \preceq v \end{array}$$

For AM9, AM10, AM11 and AM12, y is not introduced in any undischarged premise or assumption, or by w. Rules for $\preceq$ are always included with any of AM12 - AM16.[8]

Conditional systems. For the systems NB_{C_x} revert to the rules of *NB*. Then add >I and >E. As we show just below, $\not>$I and $\not>$E are derived.

$$>\!\textbf{I}\;\begin{array}{|l} \begin{array}{|l} \underline{P_{s/t}} \\ \\ Q_t \end{array} \\ (P > Q)_s \end{array} \qquad >\!\textbf{E}\;\begin{array}{|l} (P > Q)_s \\ P_{s/t} \\ \\ Q_t \end{array} \qquad \not>\!\textbf{I}\;\begin{array}{|l} P_{\overline{s}/t} \\ \neg Q_{\overline{t}} \\ \\ \neg(P > Q)_s \end{array} \qquad \not>\!\textbf{E}\;\begin{array}{|l} \neg(P > Q)_s \\ \begin{array}{|l} P_{\overline{s}/t} \\ \underline{\neg Q_{\overline{t}}} \\ \\ R_u \end{array} \\ R_u \end{array}$$

where t is not introduced in any undischarged premise or assumption

where t is not introduced in any undischarged premise or assumption, or by u

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[8] There are also rules, $\preceq^{\rho}$ according to which $\vdash s \preceq s$ and $\preceq^{\tau}$ according to which $s \preceq t$, $t \preceq u \vdash s \preceq u$. But these do not normally play a role in derivations.

As before, corresponding to constraints (1) and (2) for the C^+ system, are AMP1 and AMP2, now restricted to apply just at the normal world 0.

$$\textbf{AMP1}\ \begin{array}{|l} P_{0/t} \\ \\ P_t \end{array} \qquad\qquad \textbf{AMP2}\ \begin{array}{|l} P_0 \\ \\ P_{0/0} \end{array}$$

Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then,

NBx $\Gamma \vdash_{NBx} A$ iff there is an NBx derivation of A_0 from the members of Γ_0.

Derived rules carry over much as one would expect. Thus, e.g.,

$$\textbf{MT}\ \begin{array}{|l} (P \supset Q)_s \\ \neg Q_{\overline{s}} \\ \\ \neg P_s \end{array} \qquad \textbf{NB}\ \begin{array}{|l} (P \equiv Q)_s \\ \neg P_{\overline{s}} \\ \\ \neg Q_s \end{array} \quad \begin{array}{|l} (P \equiv Q)_s \\ \neg Q_{\overline{s}} \\ \\ \neg P_s \end{array} \qquad \textbf{DS}\ \begin{array}{|l} (P \vee Q)_s \\ \neg P_{\overline{s}} \\ \\ Q_s \end{array} \quad \begin{array}{|l} (P \vee Q)_s \\ \neg Q_{\overline{s}} \\ \\ P_s \end{array}$$

$$\textbf{Impl} \quad \begin{array}{l} (P \supset Q)_s \ \lhd\rhd\ (\neg P \vee Q)_s \\ (\neg P \supset Q)_s \ \lhd\rhd\ (P \vee Q)_s \end{array}$$

Examples. First, $\not\to$I, $\not\to$E, $\not>$I and $\not>$E are derived rules in NB_x and NB_{Cx}.

$\not\to$**I**

$$\begin{array}{rll}
1 & \vert\ \overline{s}.t.u & \text{P} \\
2 & \vert\ P_t & \text{P} \\
3 & \vert\ \underline{\neg Q_{\overline{u}}} & \text{P} \\
4 & \vert\ \vert\ \underline{(P \to Q)_{\overline{s}}} & \text{A } (c, \neg\text{I}) \\
5 & \vert\ \vert\ Q_u & \text{1,4,2} \to\text{E} \\
6 & \vert\ \vert\ \neg Q_{\overline{u}} & \text{3 R} \\
7 & \vert\ \neg(P \to Q)_s & \text{4-6 } \neg\text{I}
\end{array}$$

$\not\to$E

$$\begin{array}{rll}
1 & \vert\ \underline{\neg(P \to Q)_s} & \text{P} \\
2 & \vert\ \vert\ \underline{\neg R_{\overline{v}}} & \text{A } (c, \neg\text{E}) \\
3 & \vert\ \vert\ \vert\ \overline{s}.t.u & \text{A } (g, \to\text{I}) \\
4 & \vert\ \vert\ \vert\ \underline{P_t} & \\
5 & \vert\ \vert\ \vert\ \vert\ \underline{\neg Q_{\overline{u}}} & \text{A } (c, \neg\text{E}) \\
 & \vert\ \vert\ \vert\ \vert\ \vdots & \text{with 1,3,4,5} \\
6 & \vert\ \vert\ \vert\ \vert\ R_v & \text{as for } \not\to\text{E} \\
7 & \vert\ \vert\ \vert\ \vert\ \neg R_{\overline{v}} & \text{2 R} \\
8 & \vert\ \vert\ \vert\ Q_u & \text{5-7 } \neg\text{E} \\
9 & \vert\ \vert\ (P \to Q)_{\overline{s}} & \text{3-8} \to\text{I} \\
10 & \vert\ \vert\ \neg(P \to Q)_s & \text{1 R} \\
11 & \vert\ R_v & \text{2-10 } \neg\text{E}
\end{array}$$

$\not>$I

1	$\mid P_{\overline{s}/t}$	P
2	$\mid \neg Q_{\overline{t}}$	P
3	$\mid\ \mid (P > Q)_{\overline{s}}$	A (c, $\neg$I)
4	$\mid\ \mid Q_t$	1,3 $>$E
5	$\mid\ \mid \neg Q_{\overline{t}}$	2 R
6	$\mid \neg(P > Q)_s$	3-5 $\neg$I

$\not>$E

1	$\mid \neg(P > Q)_s$	P
2	$\mid\ \mid \neg R_{\overline{u}}$	A (c, $\neg$E)
3	$\mid\ \mid\ \mid P_{\overline{s}/t}$	A (g, $>$I)
4	$\mid\ \mid\ \mid\ \mid \neg Q_{\overline{t}}$	A (c, $\neg$E)
	$\mid\ \mid\ \mid\ \mid \vdots$	with 1,3,4
5	$\mid\ \mid\ \mid\ \mid R_u$	as for $\not>$E
6	$\mid\ \mid\ \mid\ \mid \neg R_{\overline{u}}$	2 R
7	$\mid\ \mid\ \mid Q_t$	4-6 $\neg$E
8	$\mid\ \mid (P > Q)_{\overline{s}}$	3-7 $>$I
9	$\mid\ \mid \neg(P > Q)_s$	1 R
10	$\mid R_u$	2-9 $\neg$E

Note the way overlines work (much the way slashes worked before). For $\not\to$E, note that the application of $\to$I depends on the restriction that t and u are not introduced by v; and similarly, for $\not>$E the application of $>$I depends on the restriction that t is not introduced by u.

As further examples, here are a few key results that parallel ones from Priest's text.

A3 $\vdash_{NBx} (A \wedge B) \to A$

1	$\mid\ \mid 0.1.2$	A (g, $\to$I)
2	$\mid\ \mid (A \wedge B)_1$	
3	$\mid\ \mid A_1$	2 $\wedge$E
4	$\mid\ \mid 1 \simeq 2$	1 0E
5	$\mid\ \mid A_2$	3,4 $\simeq$E
6	$\mid [(A \wedge B) \to A]_0$	1-5 $\to$I

A5 $\vdash_{NBx} [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

1	$0.1.2$	A (g, $\to$I)
2	$[(A \to B) \wedge (A \to C)]_1$	
3	$2.3.4$	A (g, $\to$I)
4	A_3	
5	$1 \simeq 2$	1 0E
6	$1.3.4$	3,5 $\simeq$E
7	$(A \to B)_1$	2 $\wedge$E
8	$(A \to C)_1$	2 $\wedge$E
9	B_4	6,7,4 $\to$E
10	C_4	6,8,4 $\to$E
11	$(B \wedge C)_4$	9,10 $\wedge$I
12	$[A \to (B \wedge C)]_2$	3-11 $\to$I
13	$([(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)])_0$	1-12 $\to$I

R5 $(A \to \neg B) \vdash_{NBx} (B \to \neg A)$

1	$(A \to \neg B)_0$	P
2	$0.1.2$	A (g, $\to$I)
3	B_1	
4	$A_{2^\#}$	A (c, $\neg$I)
5	$2^\# \simeq 2^\#$	$\simeq$I
6	$0.2^\#.2^\#$	5 0I
7	$\neg B_{2^\#}$	6,1,4 $\to$E
8	$1 \simeq 2$	2 0E
9	B_2	3,8 $\simeq$E
10	$\neg A_2$	4-9 $\neg$I
11	$(B \to \neg A)_0$	2-10 $\to$I

A9 $\vdash_{NB_9} (A \to B) \to [(B \to C) \to (A \to C)]$

1	$\mid \mid 0.1.2$	A (g, →I)
2	$\mid \mid (A \to B)_1$	
3	$\mid \mid \mid 2.3.4$	A (g, →I)
4	$\mid \mid \mid (B \to C)_3$	
5	$\mid \mid \mid \mid 4.5.6$	A (g, →I)
6	$\mid \mid \mid \mid A_5$	
7	$\mid \mid \mid \mid 1 \simeq 2$	1 0E
8	$\mid \mid \mid \mid (A \to B)_2$	2,7 $\simeq$E
9	$\mid \mid \mid \mid \mid 2.5.7$	A (g, 3,5 AM9)
10	$\mid \mid \mid \mid \mid 3.7.6$	
11	$\mid \mid \mid \mid \mid B_7$	9,8,6 →E
12	$\mid \mid \mid \mid \mid C_6$	10,4,11 →E
13	$\mid \mid \mid \mid C_6$	3,5,9-12 AM9
14	$\mid \mid \mid (A \to C)_4$	5-13 →I
15	$\mid \mid [(B \to C) \to (A \to C)]_2$	3-14 →I
16	$\mid ((A \to B) \to [(B \to C) \to (A \to C)])_0$	1-15 →I

$\vdash_{NB_R} (\neg A \to A) \to A$

1	$\mid \mid 0.1.2$	A (g, →I)
2	$\mid \mid (\neg A \to A)_1$	
3	$\mid \mid 0.2^{\#}.1^{\#}$	1 AM8
4	$\mid \mid \mid 0.2^{\#}.3$	A (g, 3 AM11)
5	$\mid \mid \mid 3.2^{\#}.1^{\#}$	
6	$\mid \mid \mid 3.1.2$	5 AM8
7	$\mid \mid \mid \mid 3 \preceq 4$	A (g, 6 AM12)
8	$\mid \mid \mid \mid 1.4.2$	
9	$\mid \mid \mid \mid \mid \neg A_{2^{\#}}$	A (c, ¬E)
10	$\mid \mid \mid \mid \mid 2^{\#} \simeq 3$	4 0E
11	$\mid \mid \mid \mid \mid \neg A_3$	9,10 $\simeq$E
12	$\mid \mid \mid \mid \mid \neg A_4$	7,11 $\preceq$E
13	$\mid \mid \mid \mid \mid A_2$	8,2,12 →E
14	$\mid \mid \mid \mid \mid \neg A_{2^{\#}}$	9R
15	$\mid \mid \mid \mid A_2$	9-14 ¬E
16	$\mid \mid \mid A_2$	6,7-15 AM12
17	$\mid \mid A_2$	3,4-16 AM11
18	$\mid [(\neg A \to A) \to A]_0$	1-17 →I

A14 $\vdash_{NB_{14}} (A \to \neg A) \to \neg A$

1	$\mid\ \mid\ 0.1.2$	A (g, $\to$I)
2	$\mid\ \mid\ \underline{(A \to \neg A)_1}$	
3	$\mid\ \mid\ 1 \simeq 2$	1 0E
4	$\mid\ \mid\ \mid\ 2 \simeq 0$	A (g, AM14)
5	$\mid\ \mid\ \mid\ \underline{2^{\#} \preceq 2}$	
6	$\mid\ \mid\ \mid\ \mid\ \underline{A_{2^\#}}$	A (c, $\neg$I)
7	$\mid\ \mid\ \mid\ \mid\ 1 \simeq 0$	3,4 $\simeq$E
8	$\mid\ \mid\ \mid\ \mid\ (A \to \neg A)_0$	2,7 $\simeq$E
9	$\mid\ \mid\ \mid\ \mid\ A_2$	5,6 $\preceq$E
10	$\mid\ \mid\ \mid\ \mid\ A_1$	9,3 $\simeq$E
11	$\mid\ \mid\ \mid\ \mid\ \neg A_2$	1,8,10 $\to$E
12	$\mid\ \mid\ \mid\ \mid\ A_{2^\#}$	6R
13	$\mid\ \mid\ \mid\ \neg A_2$	6-12 $\neg$I
14	$\mid\ \mid\ \mid\ 2 \not\simeq 0$	A (g, AM14)
15	$\mid\ \mid\ \mid\ \underline{2.2^{\#}.2}$	
16	$\mid\ \mid\ \mid\ \mid\ \underline{A_{2^\#}}$	A (c, $\neg$I)
17	$\mid\ \mid\ \mid\ \mid\ (A \to \neg A)_2$	2,3 $\simeq$E
18	$\mid\ \mid\ \mid\ \mid\ \neg A_2$	15,17,16 $\to$E
19	$\mid\ \mid\ \mid\ \mid\ A_{2^\#}$	16 R
20	$\mid\ \mid\ \mid\ \neg A_2$	16-19 $\neg$I
21	$\mid\ \mid\ \neg A_2$	4-13,14-20 AM14
22	$\mid\ [(A \to \neg A) \to \neg A]_0$	1-21 $\to$I

8.3 Soundness and Completeness

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, N, R, *, \preceq, v\rangle$ or $\langle W, N, R, \{R_A \mid A \in \Im\}, *, v\rangle$, let m be a map from subscripts into W such that $m(0) \in N$ and $m(\overline{s}) = m(s)^*$. Say $\langle W, N, R, *, \preceq, v\rangle_m$ and $\langle W, N, R, \{R_A | A \in \Im\}, *, v\rangle_m$ are$\langle W, N, R, *, \preceq, v\rangle$ and $\langle W, N, R, \{R_A| A \in \Im\}, *, v\rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $v_m(\Gamma) = 1$ iff for each $A_s \in \Gamma$, $v_{m(s)}(A) = 1$, for each $s \simeq t \in \Gamma$, $m(s) = m(t)$, for each $s.t.u \in \Gamma$, $\langle m(s), m(t), m(u)\rangle \in R$, for each $s \preceq t \in \Gamma$, $\langle m(s), m(t)\rangle \in\ \preceq$, and for each $A_{s/t} \in \Gamma$, $\langle m(s), m(t)\rangle \in R_A$. Unless otherwise noted, reasoning is meant to be neutral between interpretations of the different types. Now expand notions of validity to include subscripted formulas, and alternate expressions as indicated in double brackets.

VBx* $\Gamma \models^*_{Bx} A_s$ $\llbracket s \simeq t \,/\, s.t.u \,/\, s \preceq t \,/\, A_{s/t} \rrbracket$ iff there is no Bx interpretation with map m such that $v_m(\Gamma) = 1$ but $v_{m(s)}(A) = 0$ $\llbracket m(s) \neq m(t)$ / $\langle m(s), m(t), m(u)\rangle \notin R$ / $\langle m(s), m(t)\rangle \notin \preceq$ /$\langle m(s), m(t)\rangle \notin R_A \rrbracket$.

NBx^* $\Gamma \vdash^*_{NBx} A_s$ $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$ iff there is an *NBx* derivation of A_s $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A have subscript 0 (and so are not of the sort $s \simeq t$, $s.t.u$, $s \preceq t$, or $A_{s/t}$). For the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 8.1 *NBx is sound: If* $\Gamma \vdash_{NBx} A$ *then* $\Gamma \models_{Bx} A$.

L8.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Bx} P_s$ $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$, then $\Gamma' \models^*_{Bx} P_s$ $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Bx} P_s$ $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$, but $\Gamma' \not\models^*_{Bx} P_s$ $\llbracket s \simeq t/s.t.u/s \preceq t/A_{s/t} \rrbracket$. From the latter, by VBX*, there is some *Bx* interpretation with v and m such that $v_m(\Gamma') = 1$ but $v_{m(s)}(P) = 0$ $\llbracket m(s) \neq m(t)$ / $\langle m(s), m(t), m(u)\rangle \notin R$ / $\langle m(s), m(t)\rangle \notin \preceq$ /$\langle m(s), m(t)\rangle \notin R_A \rrbracket$. But since $v_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $v_m(\Gamma) = 1$; so $v_m(\Gamma) = 1$ but $v_{m(s)}(P) = 0$ $\llbracket m(s) \neq m(t)$ / $\langle m(s), m(t), m(u)\rangle \notin R$ / $\langle m(s), m(t)\rangle \notin \preceq$ /$\langle m(s), m(t)\rangle \notin R_A \rrbracket$; so by VBX*, $\Gamma \not\models^*_{Bx} P_s$ $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \models^*_{Bx} P_s$ $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$, then $\Gamma' \models^*_{Bx} P_s$ $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NBx} \mathcal{P}$ then $\Gamma \models^*_{Bx} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are formulas with subscript 0. Suppose $\Gamma \vdash^*_{NBx} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \models^*_{Bx} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption A_s $\llbracket s \simeq t$ / $s.t.u$ / $s \preceq t$ / $A_{s/t} \rrbracket$. Then $\Gamma_1 = \{A_s\}$ $\llbracket \{s \simeq t\}$ / $\{s.t.u\}$ / $\{s \preceq t\}$ / $\{A_{s/t}\} \rrbracket$; so for any *Bx* interpretation with its v and m, $v_m(\Gamma_1) = 1$ iff $v_{m(s)}(A) = 1$ $\llbracket m(s) = m(t)$ / $\langle m(s), m(t), m(u)\rangle \in R$ / $\langle m(s), m(t)\rangle \in R_A \rrbracket$; so there is no *Bx* interpretation with v and m such that $v_m(\Gamma_1) = 1$ but $v_{m(s)}(A) = 0$ $\llbracket m(s) \neq m(t)$ / $\langle m(s), m(t), m(u)\rangle \notin R$ / $\langle m(s), m(t)\rangle \notin$

$\preceq / \langle m(s), m(t)\rangle \notin R_A]\!]$. So by VBx*, $\Gamma_1 \models^*_{Bx} A_s\ [\![s \simeq t \,/\, s.t.u \,/\, s \preceq t \,/\, A_{s/t}]\!]$, where this is just to say, $\Gamma_1 \models^*_{Bx} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \models^*_{Bx} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, ∧I, ∧E, ∨I, ∨E, ¬I, ¬E, →I, →E, ≃I, ≃E, 0I, 0E or, depending on the system, AM8, AM9, AM10, AM11, AM12, AM13, AM14, AM15, AM16, ≼E, $\preceq^{\#}$, $\preceq^{R}$, >I, >E, AMp1, or AMp2. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

(∧I)

(∧E)

(∨I)

(∨E)

(¬I) If $\mathcal{P}_k$ arises by ¬I, then the picture is like this,

$$\begin{array}{r|l|l} & & \underline{A_{\overline{s}}} \\ & & \\ i & & B_t \\ j & & \neg B_{\overline{t}} \\ k & \multicolumn{2}{l}{\neg A_s} \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\neg A_s$. By assumption, $\Gamma_i \models^*_{Bx} B_t$ and $\Gamma_j \models^*_{Bx} \neg B_{\overline{t}}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{A_{\overline{s}}\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{A_{\overline{s}}\}$; so by L8.1, $\Gamma_k \cup \{A_{\overline{s}}\} \models^*_{Bx} B_t$ and $\Gamma_k \cup \{A_{\overline{s}}\} \models^*_{Bx} \neg B_{\overline{t}}$. Suppose $\Gamma_k \not\models^*_{Bx} \neg A_s$; then by VBx*, there is a Bx interpretation with v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(\neg A) = 0$; so by TB(¬), $v_{m(s)^*}(A) = 1$; so by the construction of m, $v_{m(\overline{s})}(A) = 1$; so $v_m(\Gamma_k) = 1$ and $v_{m(\overline{s})}(A) = 1$; so $v_m(\Gamma_k \cup \{A_{\overline{s}}\}) = 1$; so by VBx*, $v_{m(t)}(B) = 1$ and $v_{m(\overline{t})}(\neg B) = 1$; from the latter, by TB(¬), $v_{m(\overline{t})^*}(B) = 0$; so by the construction of m, $v_{m(t)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} \neg A_s$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(¬E)

($\to$I) If $\mathcal{P}_k$ arises by $\to$I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & s.t.u \\
 & & A_t \\ \cline{3-3}
 & & \\
i & & B_u \\
k & \multicolumn{2}{l}{(A \to B)_s}
\end{array}$$

where $i < k$, t, u are not introduced in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $(A \to B)_s$. By assumption, $\Gamma_i \models^*_{Bx} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s.t.u, A_t\}$; so by L8.1, $\Gamma_k \cup \{s.t.u, A_t\} \models^*_{Bx} B_u$. Suppose $\Gamma_k \not\models^*_{Bx} (A \to B)_s$; then by VBx*, there is a Bx interpretation with W, R, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(s)}(A \to B) = 0$; so by TB($\to$), there are $x, y \in W$ such that $Rm(s)xy$ and $v_x(A) = 1$ but $v_y(B) = 0$. Now consider a map m' like m except that $m'(t) = x$, $m'(\bar{t}) = x^*$, $m'(u) = y$, and $m'(\bar{u}) = y^*$; since t and u (along with $\bar{t}$ and $\bar{u}$) do not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; since $v_x(A) = 1$, $v_{m'(t)}(A) = 1$; and since $Rm(s)xy$, with $m(s) = m'(s)$, we have $\langle m'(s), m'(t), m'(u)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t.u, A_t\}) = 1$; so by VBx*, $v_{m'(u)}(B) = 1$. But $m'(u) = y$; so $v_y(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} (A \to B)_s$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

($\to$E) If $\mathcal{P}_k$ arises by $\to$E, then the picture is like this,

$$\begin{array}{r|l}
h & s.t.u \\
i & (A \to B)_s \\
j & A_t \\
 & \\
k & B_u
\end{array}$$

where $h, i, j < k$ and $\mathcal{P}_k$ is B_u. By assumption, $\Gamma_h \models^*_{Bx} s.t.u$, $\Gamma_i \models^*_{Bx} (A \to B)_s$ and $\Gamma_j \models^*_{Bx} A_t$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} s.t.u$, $\Gamma_k \models^*_{Bx} (A \to B)_s$ and $\Gamma_k \models^*_{Bx} A_t$. Suppose $\Gamma_k \not\models^*_{Bx} B_u$; then by VBx*, there is some Bx interpretation with W, R, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(u)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(u)\rangle \in R$, $v_{m(s)}(A \to B) = 1$ and $v_{m(t)}(A) = 1$; since $v_{m(s)}(A \to B) = 1$, by TB($\to$), there are no $x, y \in W$ such that $Rm(s)xy$ and $v_x(A) = 1$ but $v_y(B) = 0$; so since $\langle m(s), m(t), m(u)\rangle \in R$, it is not the case that $v_{m(t)}(A) = 1$ and $v_{m(u)}(B) = 0$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} B_u$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

($\simeq$I) If $\mathcal{P}_k$ arises by $\simeq$I, then the picture is like this,

$$\begin{array}{r|l} & \\ k & s \simeq s \end{array}$$

where $\mathcal{P}_k$ is $s \simeq s$. Suppose $\Gamma_k \nvDash^*_{\mathit{Bx}} s \simeq s$; then by VBx*, there is a *Bx* interpretation with v and m such that $v_m(\Gamma_k) = 1$ but $m(s) \neq m(s)$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{\mathit{Bx}} s \simeq s$, which is to say, $\Gamma_k \vDash^*_{\mathit{Bx}} \mathcal{P}_k$.

($\simeq$E) If A_k arises by $\simeq$E, then the picture is like this,

$$\begin{array}{r|l} i & s \simeq t \\ j & \mathcal{A}(s) \\ & \\ k & \mathcal{A}(t) \end{array} \qquad \text{or} \qquad \begin{array}{r|l} i & s \simeq t \\ j & \mathcal{A}(\overline{s}) \\ & \\ k & \mathcal{A}(\overline{t}) \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $\mathcal{A}(t)$ or $\mathcal{A}(\overline{t})$. By assumption, $\Gamma_i \vDash^*_{\mathit{Bx}} s \simeq t$ and $\Gamma_j \vDash^*_{\mathit{Bx}} \mathcal{A}(s)$ / $\mathcal{A}(\overline{s})$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \vDash^*_{\mathit{Bx}} s \simeq t$ and $\Gamma_k \vDash^*_{\mathit{Bx}} \mathcal{A}(s)$ / $\mathcal{A}(\overline{s})$. In the right-hand case, $\mathcal{A}(\overline{s})$ is of the sort, A_u, $u \simeq v$, $u.v.w$ or $A_{u/v}$ where one of u, v, or w is $\overline{s}$. Suppose $\mathcal{A}(\overline{s})$ is $A_{\overline{s}}$ and $\Gamma_k \nvDash^*_{\mathit{Bx}} A_{\overline{t}}$. Then by VBx*, there is some *Bx* interpretation with v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(\overline{t})}(A) = 0$. Since $v_m(\Gamma_k) = 1$, by VBx*, $m(s) = m(t)$ and $v_{m(\overline{s})}(A) = 1$; since $m(s) = m(t)$, $m(s)^* = m(t)^*$; but by the construction of m, $m(s)^* = m(\overline{s})$ and $m(t)^* = m(\overline{t})$; so $m(\overline{s}) = m(\overline{t})$; so $v_{m(\overline{t})}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{\mathit{Bx}} A_{\overline{t}}$, which is to say, $\Gamma_k \vDash^*_{\mathit{Bx}} \mathcal{P}_k$. And similarly in the other cases.

(0I) If $\mathcal{P}_k$ arises by 0I, then the picture is like this,

$$\begin{array}{r|l} i & s \simeq t \\ & \\ k & 0.s.t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $0.s.t$. By assumption, $\Gamma_i \vDash^*_{\mathit{Bx}} s \simeq t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \vDash^*_{\mathit{Bx}} s \simeq t$. Suppose $\Gamma_k \nvDash^*_{\mathit{Bx}} 0.s.t$; then by VBx*, there is a *Bx* interpretation with W, N, R, v and m such that $v_m(\Gamma_k) = 1$ but $\langle m(0), m(s), m(t)\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VBx*, $m(s) = m(t)$; and by the construction of m, $m(0) \in N$; so by NC, $\langle m(0), m(s), m(t)\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{\mathit{Bx}} 0.s.t$, which is to say, $\Gamma_k \vDash^*_{\mathit{Bx}} \mathcal{P}_k$.

(0E) If $\mathcal{P}_k$ arises by 0E, then the picture is like this,

$$\begin{array}{r|l} i & 0.s.t \\ & \\ k & s \simeq t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $s \simeq t$. By assumption, $\Gamma_i \models^*_{Bx} 0.s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} 0.s.t$. Suppose $\Gamma_k \not\models^*_{Bx} s \simeq t$; then by VBx*, there is a *Bx* interpretation with W, N, R, v and m such that $v_m(\Gamma_k) = 1$ but $m(s) \neq m(t)$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(0), m(s), m(t)\rangle \in R$; and by the construction of m, $m(0) \in N$; so by NC, $m(s) = m(t)$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} s \simeq t$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM8) If $\mathcal{P}_k$ arises by AM8, then the picture is like this,

$$\begin{array}{r|l} i & s.t.u \\ & \\ k & s.\overline{u}.\overline{t} \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $s.\overline{u}.\overline{t}$. Where this rule is included in *NBx*, *Bx* includes condition C8. By assumption, $\Gamma_i \models^*_{Bx} s.t.u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} s.t.u$. Suppose $\Gamma_k \not\models^*_{Bx} s.\overline{u}.\overline{t}$; then by VBx*, there is a *Bx* interpretation with R, v and m such that $v_m(\Gamma_k) = 1$ but $\langle m(s), m(\overline{u}), m(\overline{t})\rangle \notin R$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(u)\rangle \in R$; so by C8, $\langle m(s), m(u)^*, m(t)^*\rangle \in R$; so by the construction of m, $\langle m(s), m(\overline{u}), m(\overline{t})\rangle \in R$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} s.\overline{u}.\overline{t}$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM9) If $\mathcal{P}_k$ arises by AM9, then the picture is like this,

$$\begin{array}{r|ll} h & \multicolumn{2}{l}{s.t.x} \\ i & \multicolumn{2}{l}{x.u.v} \\ & \vline & s.u.y \\ & \vline & \underline{t.y.v} \\ & \vline & \\ & \vline & \\ j & \vline & A_w \\ k & \multicolumn{2}{l}{A_w} \end{array}$$

where $h, i, j < k$, y is not introduced in any member of Γ_k (in any undischarged premise or assumption) or by w, and $\mathcal{P}_k$ is A_w. Where this rule is included in *NBx*, *Bx* includes condition C9. By assumption, $\Gamma_h \models^*_{Bx} s.t.x$, $\Gamma_i \models^*_{Bx} x.u.v$ and $\Gamma_j \models^*_{Bx} A_w$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s.u.y, t.y.v\}$; so by L8.1,

$\Gamma_k \models^*_{Bx} s.t.x$, $\Gamma_k \models^*_{Bx} x.u.v$ and $\Gamma_k \cup \{s.u.y, t.y.v\} \models^*_{Bx} A_w$. Suppose $\Gamma_k \not\models^*_{Bx} A_w$; then by VBx*, there is a *Bx* interpretation with *W*, *R*, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(w)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(x)\rangle \in R$ and $\langle m(x), m(u), m(v)\rangle \in R$; so by C9, there is some $z \in W$ such that $\langle m(s), m(u), z\rangle \in R$ and $\langle m(t), z, m(v)\rangle \in R$. Now consider a map m' like m except that $m'(y) = z$ and $m'(\overline{y}) = z^*$; since y (along with $\overline{y}$) does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m(s) = m'(s)$, and similarly for t, u and v, $\langle m'(s), m'(u), m'(y)\rangle \in R$ and $\langle m'(t), m'(y), m'(v)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.u.y, t.y.v\}) = 1$; so by VBx*, $v_{m'(w)}(A) = 1$. But since y is not introduced by w, $m'(w) = m(w)$; so $v_{m(w)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_w$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM10)

(AM11) If $\mathcal{P}_k$ arises by AM11, then the picture is like this,

$$\begin{array}{r|l}
i & s.t.u \\
 & \vert\; s.t.y \\
 & \vert\; \underline{y.t.u} \\
 & \vert \\
 & \vert \\
j & \vert\; A_w \\
k & A_w
\end{array}$$

where $i, j < k$, y is not introduced in any member of Γ_k (in any undischarged premise or assumption) or by w, and $\mathcal{P}_k$ is A_w. Where this rule is included in *NBx*, *Bx* includes condition C11. By assumption, $\Gamma_i \models^*_{Bx} s.t.u$ and $\Gamma_j \models^*_{Bx} A_w$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s.t.y, y.t.u\}$; so by L8.1, $\Gamma_k \models^*_{Bx} s.t.u$ and $\Gamma_k \cup \{s.t.y, y.t.u\} \models^*_{Bx} A_w$. Suppose $\Gamma_k \not\models^*_{Bx} A_w$; then by VBx*, there is a *Bx* interpretation with *W*, *R*, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(w)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(u)\rangle \in R$; so by C11, there is some $z \in W$ such that $\langle m(s), m(t), z\rangle \in R$ and $\langle z, m(t), m(u)\rangle \in R$. Now consider a map m' like m except that $m'(y) = z$ and $m'(\overline{y}) = z^*$; since y (along with $\overline{y}$) does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m(s) = m'(s)$, and similarly for t and u, $\langle m'(s), m'(t), m'(y)\rangle \in R$ and $\langle m'(y), m'(t), m'(u)\rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s.t.y, y.t.u\}) = 1$; so by VBx*, $v_{m'(w)}(A) = 1$. But since y is not introduced by w, $m'(w) = m(w)$; so $v_{m(w)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_w$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM12) If $\mathcal{P}_k$ arises by AM12, then the picture is like this,

$$
\begin{array}{r|l|l}
i & \multicolumn{2}{l}{s.t.u} \\
 & & s \preceq y \\
 & & \underline{t.y.u} \\
 & & \\
j & & A_w \\
k & \multicolumn{2}{l}{A_w}
\end{array}
$$

where $i, j < k$, y is not introduced in any member of Γ_k (in any undischarged premise or assumption) or by w, and $\mathcal{P}_k$ is A_w. Where this rule is included in *NBx*, *Bx* includes condition C12. By assumption, $\Gamma_i \models^*_{Bx} s.t.u$ and $\Gamma_j \models^*_{Bx} A_w$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s \preceq y, t.y.u\}$; so by L8.1, $\Gamma_k \models^*_{Bx} s.t.u$ and $\Gamma_k \cup \{s \preceq y, t.y.u\} \models^*_{Bx} A_w$. Suppose $\Gamma_k \not\models^*_{Bx} A_w$; then by VBx*, there is a *Bx* interpretation with *W*, *R*, $\preceq$, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(w)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t), m(u) \rangle \in R$; so by C12, there is some z such that $\langle m(s), z \rangle \in \preceq$ and $\langle m(t), z, m(u) \rangle \in R$. Now consider a map m' like m except that $m'(y) = z$ and $m'(\overline{y}) = z^*$; since y (along with $\overline{y}$) does not appear in Γ_k, it remains that $v_{m'}(\Gamma_k) = 1$; and since $m(s) = m'(s)$, and similarly for t and u, $\langle m'(s), m'(y) \rangle \in \preceq$ and $\langle m'(t), m'(y), m'(u) \rangle \in R$; so $v_{m'}(\Gamma_k \cup \{s \preceq y, t.y.u\}) = 1$; so by VBx*, $v_{m'(w)}(A) = 1$. But since y is not introduced by w, $m'(w) = m(w)$; so $v_{m(w)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_w$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM13) If $\mathcal{P}_k$ arises by AM13, then the picture is like this,

$$
\begin{array}{r|l}
 & \\
 & \\
 & \\
k & 0^{\#} \preceq 0
\end{array}
$$

where $\mathcal{P}_k$ is $0^{\#} \preceq 0$. Where this rule is included in *NBx*, *Bx* includes condition C13. Suppose $\Gamma_k \not\models^*_{Bx} 0^{\#} \preceq 0$; then by VBx*, there is a *Bx* interpretation with *W*, *N*, *R*, $\preceq$, v and m such that $v_m(\Gamma_k) = 1$ but $\langle m(0)^*, m(0) \rangle \notin \preceq$. But by the construction of m, $m(0) \in N$; so by C13, $\langle m(0)^*, m(0) \rangle \in \preceq$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} 0^{\#} \preceq 0$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM14) If $\mathcal{P}_k$ arises by AM14, then the picture is like this,

$$
\begin{array}{r|c|l}
 & & s \simeq 0 \\
 & & \overline{s} \preceq s \\ \cline{3-3}
 & & \\
i & & A_w \\
 & \multicolumn{2}{l}{} \\
 & & s \not\simeq 0 \\
 & & s.\overline{s}.s \\ \cline{3-3}
 & & \\
j & & A_w \\
k & \multicolumn{2}{l}{A_w}
\end{array}
$$

where $i, j < k$ and $\mathcal{P}_k$ is A_w. Where this rule is included in *NBx*, *Bx* includes condition C14. By assumption, $\Gamma_i \models^*_{Bx} A_w$ and $\Gamma_j \models^*_{Bx} A_w$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{s \simeq 0, \overline{s} \preceq s\}$ and $\Gamma_j \subseteq \Gamma_k \cup \{s \not\simeq 0, s.\overline{s}.s\}$; so by L8.1, $\Gamma_k \cup \{s \simeq 0, \overline{s} \preceq s\} \models^*_{Bx} A_w$ and $\Gamma_k \cup \{s \not\simeq 0, s.\overline{s}.s\} \models^*_{Bx} A_w$. Suppose $\Gamma_k \not\models^*_{Bx} A_w$; then by VBx*, there is a *Bx* interpretation with W, N, R, $\preceq$, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(w)}(A) = 0$. Consider world $m(s) \in W$; either $m(s) \in N$ or $m(s) \notin N$. Suppose $m(s) \in N$; then by the construction of m, $m(s) = m(0)$, and by C14, $\langle m(s)^*, m(s)\rangle \in \preceq$; so by the construction of m, $\langle m(\overline{s}), m(s)\rangle \in \preceq$; so $v_m(\Gamma_k \cup \{s \simeq 0, \overline{s} \preceq s\}) = 1$; so by VBx*, $v_{m(w)}(A) = 1$. Suppose $m(s) \notin N$; then by the construction of m, $m(s) \neq m(0)$, and by C14, $\langle m(s), m(s)^*, m(s)\rangle \in R$; so by the construction of m, $\langle m(s), m(\overline{s}), m(s)\rangle \in R$; so $\Gamma_k \cup \{s \not\simeq 0, s.\overline{s}.s\} = 1$; so by VBx*, $v_{m(w)}(A) = 1$; so in either case, $v_{m(w)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_w$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

(AM15)

(AM16)

($\preceq$E) If $\mathcal{P}_k$ arises by $\preceq$E, then the picture is like this,

$$
\begin{array}{r|l}
i & s \preceq t \\
j & A_s \\
 & \\
k & A_t
\end{array}
$$

where $i, j < k$ and $\mathcal{P}_k$ is A_t. By assumption, $\Gamma_i \models^*_{Bx} s \preceq t$ and $\Gamma_j \models^*_{Bx} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} s \preceq t$ and $\Gamma_k \models^*_{Bx} A_s$. Suppose $\Gamma_k \not\models^*_{Bx} A_t$; then by VBx*, there is a *Bx* interpretation with W, N, R, $\preceq$, v and m

such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t)\rangle \in \preceq$ and $v_{m(s)}(A) = 1$.

Now, by induction on the number of operators in A, we show that for any x, $y \in W$, if $x \preceq y$, then (i) if $v_x(A) = 1$ then $v_y(A) = 1$ and (ii) if $v_{y^*}(A) = 1$ then $v_{x^*}(A) = 1$.

Basis: A is a parameter p. Suppose $x \preceq y$. Then $x \trianglelefteq y$ and $y^* \trianglelefteq x^*$. (i) Suppose $v_x(A) = 1$; then $v_x(p) = 1$; and since $x \trianglelefteq y$, $v_y(p) = 1$, which is to say $v_y(A) = 1$. (ii) Suppose $v_{y^*}(A) = 1$; then $v_{y^*}(p) = 1$; and since $y^* \trianglelefteq x^*$, $v_{x^*}(p) = 1$, which is to say $v_{x^*}(A) = 1$.

Assp: For any i, $0 \leq i < k$, if A has i operators, then for any x, $y \in W$, if $x \preceq y$, then (i) if $v_x(A) = 1$ then $v_y(A) = 1$ and (ii) if $v_{y^*}(A) = 1$ then $v_{x^*}(A) = 1$.

Show: If A has k operators, then for any x, $y \in W$, if $x \preceq y$, then (i) if $v_x(A) = 1$ then $v_y(A) = 1$ and (ii) if $v_{y^*}(A) = 1$ then $v_{x^*}(A) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \rightarrow Q$, where P and Q have $< k$ operators. Suppose $x \preceq y$. Then $x \trianglelefteq y$ and $y^* \trianglelefteq x^*$.

($\neg$) A is $\neg P$. (i) Suppose $v_x(A) = 1$; then $v_x(\neg P) = 1$; so by HBx($\neg$), $v_{x^*}(P) = 0$; so by assumption, $v_{y^*}(P) = 0$; so by HBx($\neg$), $v_y(\neg P) = 1$, which is to say, $v_y(A) = 1$. (ii) Suppose $v_{y^*}(A) = 1$; then $v_{y^*}(\neg P) = 1$; so by HBx($\neg$), $v_y(P) = 0$; so by assumption, $v_x(P) = 0$; so by HBx($\neg$), $v_{x^*}(\neg P) = 1$, which is to say, $v_{x^*}(A) = 1$.

($\wedge$) A is $P \wedge Q$. (i) Suppose $v_x(A) = 1$; then $v_x(P \wedge Q) = 1$; so by HBx($\wedge$), $v_x(P) = 1$ and $v_x(Q) = 1$; so by assumption, $v_y(P) = 1$ and $v_y(Q) = 1$; so by HBx($\wedge$), $v_y(P \wedge Q) = 1$, which is to say $v_y(A) = 1$. (ii) Suppose $v_{y^*}(A) = 1$; then $v_{y^*}(P \wedge Q) = 1$; so by HBx($\wedge$), $v_{y^*}(P) = 1$ and $v_{y^*}(Q) = 1$; so by assumption, $v_{x^*}(P) = 1$ and $v_{x^*}(Q) = 1$; so by HBx($\wedge$), $v_{x^*}(P \wedge Q) = 1$, which is to say $v_{x^*}(A) = 1$.

($\vee$)

($\rightarrow$) A is $P \rightarrow Q$. (i) Suppose $v_x(A) = 1$ but $v_y(A) = 0$; then $v_x(P \rightarrow Q) = 1$ and $v_y(P \rightarrow Q) = 0$; then by HBx($\rightarrow$), there are some w, $z \in W$ such that $yRwz$ and $v_w(P) = 1$ but $v_z(Q) = 0$. We consider this in two cases: (1) $x \notin N$; then since

$x \trianglelefteq y$ and $yRwz$, $xRwz$; so since $v_w(P) = 1$, $v_x(P \rightarrow Q) = 1$, and $xRwz$, by HBx($\rightarrow$), $v_z(Q) = 1$. This is impossible. Case (2): $x \in N$; then since $x \trianglelefteq y$ and $yRwz$, $w \preceq z$; so since $x \in N$ and $w = w$, by NC, $xRww$; so since $v_w(P) = 1$, $v_x(P \rightarrow Q) = 1$, and $xRww$, by HBx($\rightarrow$), $v_w(Q) = 1$; but since $w \preceq z$, by assumption, $v_z(Q) = 1$. This is impossible; reject the assumption: $v_y(P \rightarrow Q) = 1$, which is to say $v_y(A) = 1$.

And similarly for (ii).

For any A and any $x, y \in W$, if $x \preceq y$, then (i) if $v_x(A) = 1$ then $v_y(A) = 1$ and (ii) if $v_{y^*}(A) = 1$ then $v_{x^*}(A) = 1$.

So, returning to the case for ($\preceq$E), $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} A_t$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

($\preceq^{\#}$) If $\mathcal{P}_k$ arises by $\preceq^{\#}$, then the picture is like this,

$$\begin{array}{r|l} i & s \preceq t \\ & \\ & \\ k & \bar{t} \preceq \bar{s} \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $\bar{t} \preceq \bar{s}$. By assumption, $\Gamma_i \models^*_{Bx} s \preceq t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{Bx} s \preceq t$. Suppose $\Gamma_k \not\models^*_{Bx} \bar{t} \preceq \bar{s}$; then by VBx*, there is an Bx interpretation with W, $\preceq$, v and m such that $v_m(\Gamma_k) = 1$ but $\langle m(\bar{t}), m(\bar{s}) \rangle \notin \preceq$; so by the construction of m, $\langle m(t)^*, m(s)^* \rangle \notin \preceq$; since $v_m(\Gamma_k) = 1$, by VBx*, $\langle m(s), m(t) \rangle \in \preceq$; so $\langle m(s), m(t) \rangle \in \trianglelefteq$ and $\langle m(t)^*, m(s)^* \rangle \in \trianglelefteq$; but since $m(s)^{**} = m(s)$ and $m(t)^{**} = m(t)$, $\langle m(s)^{**}, m(t)^{**} \rangle \in \trianglelefteq$; so $\langle m(t)^*, m(s)^* \rangle \in \preceq$. This is impossible; reject the assumption: $\Gamma_k \models^*_{Bx} \bar{t} \preceq \bar{s}$, which is to say, $\Gamma_k \models^*_{Bx} \mathcal{P}_k$.

($\preceq^{R}$)

($>$I)

($>$E) If $\mathcal{P}_k$ arises by $>$E, then the picture is like this,

$$\begin{array}{r|l} i & (A > B)_s \\ j & A_{s/t} \\ & \\ k & B_t \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is B_t. By assumption, $\Gamma_i \models^*_{Bx} (A > B)_s$ and $\Gamma_j \models^*_{Bx} A_{s/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by

L8.1, $\Gamma_k \models^*_{\mathit{Bx}} (A > B)_s$ and $\Gamma_k \models^*_{\mathit{Bx}} A_{s/t}$. Suppose $\Gamma_k \not\models^*_{\mathit{Bx}} B_t$; then by VBx*, there is some *Bx* interpretation with W, $\{R_A \mid A \in \Im\}$, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(B) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $v_{m(s)}(A > B) = 1$ and $\langle m(s), m(t)\rangle \in R_A$; from the former, by TB(>), any $w \in W$ such that $m(s)R_A w$ has $v_w(B) = 1$; so $v_{m(t)}(B) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\mathit{Bx}} B_t$, which is to say, $\Gamma_k \models^*_{\mathit{Bx}} \mathcal{P}_k$.

(AMp1) If $\mathcal{P}_k$ arises by AMp1, then the picture is like this,

$$\begin{array}{r|l} i & A_{0/t} \\ & \\ k & A_t \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is A_t. Where this rule is in *NBx*, *Bx* includes condition (1). By assumption, $\Gamma_i \models^*_{\mathit{Bx}} A_{0/t}$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{\mathit{Bx}} A_{0/t}$. Suppose $\Gamma_k \not\models^*_{\mathit{Bx}} A_t$; then by VBx*, there is some *Bx* interpretation with N, $\{R_A \mid A \in \Im\}$, v and m such that $v_m(\Gamma_k) = 1$ but $v_{m(t)}(A) = 0$; since $v_m(\Gamma_k) = 1$, by VBx*, $m(t) \in f_A(m(0))$; but by the construction of m, $m(0) \in N$; so by condition (1), $m(t) \in [A]$; so $v_{m(t)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\mathit{Bx}} A_t$, which is to say, $\Gamma_k \models^*_{\mathit{Bx}} \mathcal{P}_k$.

(AMp2) If $\mathcal{P}_k$ arises by AMp2, then the picture is like this,

$$\begin{array}{r|l} i & A_0 \\ & \\ k & A_{0/0} \end{array}$$

where $i < k$ and $\mathcal{P}_k$ is $A_{0/0}$. Where this rule is in *NBx*, *Bx* includes condition (2). By assumption, $\Gamma_i \models^*_{\mathit{Bx}} A_0$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L8.1, $\Gamma_k \models^*_{\mathit{Bx}} A_0$. Suppose $\Gamma_k \not\models^*_{\mathit{Bx}} A_{0/0}$; then by VBx*, there is some *Bx* interpretation with N, $\{R_A \mid A \in \Im\}$, v and m such that $v_m(\Gamma_k) = 1$ but $m(0) \notin f_A(m(0))$; since $v_m(\Gamma_k) = 1$, by VCx*, $v_{m(0)}(A) = 1$; so $m(0) \in [A]$; and by the construction of m, $m(0) \in N$; so by condition (2), $m(0) \in f_A(m(0))$. This is impossible; reject the assumption: $\Gamma_k \models^*_{\mathit{Bx}} A_{0/0}$, which is to say, $\Gamma_k \models^*_{\mathit{Bx}} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{\mathit{Bx}} \mathcal{P}_i$.

THEOREM 8.2 *NBx is complete: if* $\Gamma \models_{\mathit{Bx}} A$ *then* $\Gamma \vdash_{\mathit{NBx}} A$.

Suppose $\Gamma \models_{Bx} A$; then $\Gamma_0 \models^*_{Bx} A_0$; we show that $\Gamma_0 \vdash^*_{NBx} A_0$. As usual, this reduces to the standard notion. For the following, fix on some particular constraint(s) x. Then definitions of *consistency* etc. are relative to it.

Con Γ is consistent iff there is no A_s such that $\Gamma \vdash^*_{NBx} A_s$ and $\Gamma \vdash^*_{NBx} \neg A_{\overline{s}}$.

L8.2 If s is 0 or introduced in Γ, and $\Gamma \nvdash^*_{NBx} \neg P_{\overline{s}}$, then $\Gamma \cup \{P_s\}$ is consistent.

Suppose s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NBx} \neg P_{\overline{s}}$ but $\Gamma \cup \{P_s\}$ is inconsistent. Then there is some A_t such that $\Gamma \cup \{P_s\} \vdash^*_{NBx} A_t$ and $\Gamma \cup \{P_s\} \vdash^*_{NBx} \neg A_{\overline{t}}$. But then we can argue,

$$\begin{array}{r|lll}
1 & \Gamma & & \\
2 & & \mid P_s & \text{A } (c, \neg\text{I}) \\
3 & & \mid A_t & \text{from } \Gamma \cup \{P_s\} \\
4 & & \mid \neg A_{\overline{t}} & \text{from } \Gamma \cup \{P_s\} \\
5 & \neg P_{\overline{s}} & & \text{2-4 } \neg\text{I}
\end{array}$$

where the assumption is allowed insofar as s is either 0 or introduced in Γ; so $\Gamma \vdash^*_{NBx} \neg P_{\overline{s}}$. But this is impossible; reject the assumption: if s is 0 or introduced in Γ and $\Gamma \nvdash^*_{NBx} \neg P_{\overline{s}}$, then $\Gamma \cup \{P_s\}$ is consistent.

L8.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1$ $\mathcal{P}_2 \ldots$ In addition, there is an enumeration of these formulas with access relations $s.t.u$ and with pairs of the sort $s.t.u$ / $u.v.w$.

Proof by construction.

Max Γ is s-maximal iff for any A_s either $\Gamma \vdash^*_{NBx} A_s$ or $\Gamma \vdash^*_{NBx} \neg A_{\overline{s}}$.

Sgt Γ is a scapegoat set for $\rightarrow$ iff for every formula of the form $\neg(A \rightarrow B)_s$, if $\Gamma \vdash^*_{NBx} \neg(A \rightarrow B)_s$ then there are y and z such that $\Gamma \vdash^*_{NBx} \overline{s}.y.z$, $\Gamma \vdash^*_{NBx} A_y$ and $\Gamma \vdash^*_{NBx} \neg B_{\overline{z}}$.

Γ is a scapegoat set for $>$ iff for every formula of the form $\neg(A > B)_s$, if $\Gamma \vdash^*_{NBx} \neg(A > B)_s$ then there is some y such that $\Gamma \vdash^*_{NBx} A_{\overline{s}/y}$ and $\Gamma \vdash^*_{NBx} \neg B_{\overline{y}}$.

Γ is a scapegoat set for C9/C10 iff for any access pair $s.t.u/u.v.w$, if $\Gamma \vdash^*_{NBx} s.t.u$ and $\Gamma \vdash^*_{NBx} u.v.w$, then there is a y such that $\Gamma \vdash^*_{NBx} s.v.y$ and $\Gamma \vdash^*_{NBx} t.y.w$, and a z such that $\Gamma \vdash^*_{NBx} t.v.z$ and $\Gamma \vdash^*_{NBx} s.z.w$.

Γ is a scapegoat set for C12 iff for any access relation $s.t.u$, if $\Gamma \vdash^*_{NBx} s.t.u$, then there is a y such that $\Gamma \vdash^*_{NBx} s.t.y$ and $\Gamma \vdash^*_{NBx} y.t.u$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L8.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the subscripted formulas, together with all the access relations $s.t.u$ if C12 is in *Bx*, and pairs $s.t.u \,/\, u.v.w$ if C9 and C10 are in *Bx*; let $\mathcal{E}_0$ be this enumeration. Then for the first expression $\mathcal{P}$ in $\mathcal{E}_{i-1}$ such that all its subscripts are 0 or introduced in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $\mathcal{P}$, and set,

$$\begin{array}{ll}
\Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \cup \{\mathcal{P}\} \text{ is inconsistent} \\
\Omega_{i^*} = \Omega_{i-1} \cup \{\mathcal{P}\} & \text{if } \Omega_{i-1} \cup \{\mathcal{P}\} \text{ is consistent}
\end{array}$$

and

$$\begin{array}{ll}
\Omega_i = \Omega_{i^*} & \text{if } \mathcal{P} \text{ is not of the form } \neg(P \rightarrow Q)_s,\ \neg(P > Q)_s,\ s.t.u/u.v.w, \text{ or } s.t.u \\
\Omega_i = \Omega_{i^*} \cup \{\overline{s}.y.z, P_y \neg Q_{\overline{z}}\} & \text{if } \mathcal{P} \text{ is of the form } \neg(P \rightarrow Q)_s \\
\Omega_i = \Omega_{i^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\} & \text{if } \mathcal{P} \text{ is of the form } \neg(P > Q)_s \\
\Omega_i = \Omega_{i^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\} & \text{if } \mathcal{P} \text{ is of the form } s.t.u/u.v.w \\
\Omega_i = \Omega_{i^*} \cup \{s.t.y, y.t.u\} & \text{if } \mathcal{P} \text{ is of the form } s.t.u
\end{array}$$

-where y and z are the first subscripts not introduced in Ω_{i^*}

then

$$\Gamma' = \bigcup_{i \geq 0} \Omega_i$$

Note that there are always sure to be subscripts y and z not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many expressions are added – the only subscripts in the initial Ω_0 being 0. Suppose s is introduced in Γ'; then there is some Ω_i in which it is first introduced; and any expression $\mathcal{P}_j$ in the original enumeration that introduces subscript s is sure to be "considered" for inclusion at a subsequent stage.

L8.4 For any s introduced in Γ', Γ' is s-maximal.

Suppose s is introduced in Γ' but Γ' is not s-maximal. Then there is some A_s such that $\Gamma' \nvdash^*_{NBx} A_s$ and $\Gamma' \nvdash^*_{NBx} \neg A_{\overline{s}}$. For any i, each member of Ω_{i-1} is in Γ'; so if $\Omega_{i-1} \vdash^*_{NBx} \neg A_{\overline{s}}$ then $\Gamma' \vdash^*_{NBx} \neg A_{\overline{s}}$; but $\Gamma' \nvdash^*_{NBx} \neg A_{\overline{s}}$; so $\Omega_{i-1} \nvdash^*_{NBx} \neg A_{\overline{s}}$; so since s is introduced in Γ', by L8.2, $\Gamma' \cup \{A_s\}$ is consistent; so there is a stage in the construction that sets $\Omega_{i^*} = \Omega_{i-1} \cup \{A_s\}$; so by construction, $A_s \in \Gamma'$; so $\Gamma' \vdash^*_{NBx} A_s$. This is impossible; reject the assumption: Γ' is s-maximal.

L8.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$, (iii) $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$, (iv) $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$, (v) $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$, or (vi) $\Omega_{k^*} \cup \{s.t.y, y.t.u\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$. Then by construction, $\Omega_{k-1} \cup \{\mathcal{P}\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg(P \rightarrow Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_x and $\neg A_{\overline{x}}$ such that $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\} \vdash^*_{NBx} A_x$ and $\Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\} \vdash^*_{NBx} \neg A_{\overline{x}}$. So reason as follows,

$$\begin{array}{r|l l}
1 & \Omega_{k^*} & \\
2 & \quad \mid \overline{s}.y.z & \text{A } (g, \rightarrow\text{I}) \\
3 & \quad \mid \underline{P_y} & \\
4 & \quad \mid \quad \mid \underline{\neg Q_{\overline{z}}} & \text{A } (c, \neg\text{E}) \\
5 & \quad \mid \quad \mid A_x & \text{from } \Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\} \\
6 & \quad \mid \quad \mid \neg A_{\overline{x}} & \text{from } \Omega_{k^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\} \\
7 & \quad \mid Q_z & \text{4-6 } \neg\text{E} \\
8 & (P \rightarrow Q)_{\overline{s}} & \text{2-7 } \rightarrow\text{I}
\end{array}$$

where, by construction, y and z are not introduced Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NBx} (P \rightarrow Q)_{\overline{s}}$; but $\neg(P \rightarrow Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NBx} \neg(P \rightarrow Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$. In this case, as above, Ω_{k*} is consistent and by construction, $\neg(P > Q)_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_x and $\neg A_{\overline{x}}$ such that $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\} \vdash^*_{NBx} A_x$ and $\Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\} \vdash^*_{NBx} \neg A_{\overline{x}}$. So reason as follows,

$$\begin{array}{r|l l}
1 & \Omega_{k^*} & \\
2 & \quad \mid \underline{P_{\overline{s}/y}} & \text{A } (g, >\text{I}) \\
3 & \quad \mid \quad \mid \underline{\neg Q_{\overline{y}}} & \text{A } (c, \neg\text{E}) \\
4 & \quad \mid \quad \mid A_x & \text{from } \Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\} \\
5 & \quad \mid \quad \mid \neg A_{\overline{x}} & \text{from } \Omega_{k^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\} \\
6 & \quad \mid Q_y & \text{3-5 } \neg\text{E} \\
8 & (P > Q)_{\overline{s}} & \text{2-6 } >\text{I}
\end{array}$$

where, by construction, y is not introduced Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NBx} (P > Q)_{\overline{s}}$; but $\neg(P > Q)_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NBx} \neg(P > Q)_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(v) Suppose Ω_k is $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$. In this case, as above, Ω_{k*} is consistent and by construction, $s.t.u, u.v.w \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are A_x and $\neg A_{\overline{x}}$ such that $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\} \vdash^*_{NBx} A_x$ and $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\} \vdash^*_{NBx} \neg A_{\overline{x}}$. So reason as follows,

1	$\vert\ \Omega_{k^*}$	
2	$\vert\ s.t.u$	member of Ω_{k^*}
3	$\vert\ u.v.w$	member of Ω_{k^*}
4	$\vert\ \vert\ s.v.y$	A (g, AM9)
5	$\vert\ \vert\ \underline{t.y.w}$	
6	$\vert\ \vert\ \vert\ t.v.z$	A (g, AM10)
7	$\vert\ \vert\ \vert\ \underline{s.z.w}$	
8	$\vert\ \vert\ \vert\ \vert\ \underline{(A \to A)_0}$	A (c, $\neg$I)
9	$\vert\ \vert\ \vert\ \vert\ A_x$	from $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$
10	$\vert\ \vert\ \vert\ \vert\ \neg A_{\overline{x}}$	from $\Omega_{k^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$
11	$\vert\ \vert\ \vert\ \neg(A \to A)_{0\#}$	8-10 $\neg$I
12	$\vert\ \vert\ \neg(A \to A)_{0\#}$	2,3,6-11 AM10
13	$\vert\ \neg(A \to A)_{0\#}$	2,3,4-12 AM9

where, by construction, y and z are not introduced Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NBx} \neg(A \to A)_{0\#}$; but $\vdash^*_{NBx} (A \to A)_0$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(vi) Similarly.

For any i, Ω_i is consistent.

L8.6 If Γ_0 is consistent, then Γ' is consistent.

Suppose Γ_0 is consistent, but Γ' is not; from the latter, there is some P_s such that $\Gamma' \vdash^*_{NBx} P_s$ and $\Gamma' \vdash^*_{NBx} \neg P_{\overline{s}}$. Consider derivations D1 and D2 of these results, and the premises $\mathcal{P}_i \ldots \mathcal{P}_j$ of these derivations. By construction, there is an Ω_k with each of these premises as a member; so D1 and D2 are derivations from Ω_k; so Ω_k is not consistent. But since Γ_0 is consistent, by L8.5, Ω_k is consistent. This is impossible; reject the assumption: if Γ_0 is consistent then Γ' is consistent.

L8.7 If Γ_0 is consistent, then Γ' is a scapegoat set for $\rightarrow$, $>$ and, in the appropriate systems, for C9/C10 and C12.

For $\rightarrow$. Suppose Γ_0 is consistent and $\Gamma' \vdash^{*}_{\textit{NBx}} \neg(P \rightarrow Q)_s$. By L8.6, Γ' is consistent; and by the constraints on subscripts, s is introduced in Γ'. Since $\Gamma' \vdash^{*}_{\textit{NBx}} \neg(P \rightarrow Q)_s$, Γ' has just the same consequences as $\Gamma' \cup \{\neg(P \rightarrow Q)_s\}$; so $\Gamma' \cup \{\neg(P \rightarrow Q)_s\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{\neg(P \rightarrow Q)_s\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg(P \rightarrow Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{\overline{s}.y.z, P_y, \neg Q_{\overline{z}}\}$; so by construction, $\overline{s}.y.z, P_y, \neg Q_{\overline{z}} \in \Gamma'$; so $\Gamma' \vdash^{*}_{\textit{NBx}} \overline{s}.y.z$, $\Gamma' \vdash^{*}_{\textit{NBx}} P_y$ and $\Gamma' \vdash^{*}_{\textit{NBx}} \neg Q_{\overline{z}}$. So Γ' is a scapegoat set for $\rightarrow$.

For $>$. Suppose Γ_0 is consistent and $\Gamma' \vdash^{*}_{\textit{NBx}} \neg(P > Q)_s$. By L8.6, Γ' is consistent; and by the constraints on subscripts, s is introduced in Γ'. Since $\Gamma' \vdash^{*}_{\textit{NBx}} \neg(P > Q)_s$, Γ' has just the same consequences as $\Gamma' \cup \{\neg(P > Q)_s\}$; so $\Gamma' \cup \{\neg(P > Q)_s\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{\neg(P > Q)_s\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{\neg(P > Q)_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{P_{\overline{s}/y}, \neg Q_{\overline{y}}\}$; so by construction, $P_{\overline{s}/y}, \neg Q_{\overline{y}} \in \Gamma'$; so $\Gamma' \vdash^{*}_{\textit{NBx}} P_{\overline{s}/y}$ and $\Gamma' \vdash^{*}_{\textit{NBx}} \neg Q_{\overline{y}}$. So Γ' is a scapegoat set for $>$.

For C9/C10. Suppose Γ_0 is consistent, $\Gamma' \vdash^{*}_{\textit{NBx}} s.t.u$ and $\Gamma' \vdash^{*}_{\textit{NBx}} u.v.w$. By L8.6, Γ' is consistent; and by the constraints on subscripts, s, t, u, v and w are introduced in Γ'. Since $\Gamma' \vdash^{*}_{\textit{NBx}} s.t.u$, and $\Gamma' \vdash^{*}_{\textit{NBx}} u.v.w$, Γ' has just the same consequences as $\Gamma' \cup \{s.t.u, u.v.w\}$; so $\Gamma' \cup \{s.t.u, u.v.w\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{s.t.u, u.v.w\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{s.t.u, u.v.w\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.v.y, t.y.w, t.v.z, s.z.w\}$; so by construction, $s.v.y, t.y.w, t.v.z, s.z.w \in \Gamma'$; so there is a y such that $\Gamma' \vdash^{*}_{\textit{NBx}} s.v.y$ and $\Gamma' \vdash^{*}_{\textit{NBx}} t.y.w$, and there is a z such that $\Gamma' \vdash^{*}_{\textit{NBx}} t.v.z$ and $\Gamma' \vdash^{*}_{\textit{NBx}} s.z.w$. So Γ' is a scapegoat set for C9/C10. And similarly for C12.

C(I) We construct an interpretation $I_{Bx} = \langle W, N, R, *, v\rangle$ or $\langle W, N, R, \{R_A \mid A \in \Im\}, *, v\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s introduced in Γ', except that if $\Gamma' \vdash^{*}_{\textit{NBx}} s \simeq t$ then $w_s = w_t$ and $w_{\overline{s}} = w_{\overline{t}}$ (we might do this, in the usual way, by beginning with equivalence classes on subscripts). Then set $N = \{w_0\}$; $\langle w_s, w_t, w_u\rangle \in R$ iff $\Gamma' \vdash^{*}_{\textit{NBx}} s.t.u$; $\langle w_s, w_t\rangle \in R_A$ iff $\Gamma' \vdash^{*}_{\textit{NBx}} A_{s/t}$; $* = \{\langle w_s, w_{\overline{s}}\rangle \mid s \text{ is introduced in } \Gamma'\}$; and $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^{*}_{\textit{NBx}} p_s$.

Note that the specification is consistent: Suppose $w_s = w_t$; then by construction, $\Gamma' \vdash^*_{NBx} s \simeq t$; so by $\simeq$E, $\Gamma' \vdash^*_{NBx} p_s$ iff $\Gamma' \vdash^*_{NBx} p_t$; so $v_{w_s}(p) = v_{w_t}(p)$; and similarly in other cases. Also, the $*$-function has the right form, as s, $\overline{s}$ are introduced in pairs, and $\langle w_s, w_{s\#} \rangle \in *$ iff $\langle w_{s\#}, w_s \rangle \in *$.

L8.8 If Γ_0 is consistent then for I_{Bx} constructed as above, and for any s introduced in Γ', $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

Suppose Γ_0 is consistent and s is introduced in Γ'. By L8.4, Γ' is s-maximal. By L8.6 and L8.7, Γ' is consistent and a scapegoat set for $\rightarrow$ and $>$. Now by induction on the number of operators in A_s,

Basis: If A_s has no operators, then it is a parameter p_s and by construction, $v_{w_s}(p) = 1$ iff $\Gamma' \vdash^*_{NBx} p_s$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

Assp: For any i, $0 \leq i < k$, if A_s has i operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

Show: If A_s has k operators, then $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

If A_s has k operators, then it is of the form $\neg P_s$, $(P \wedge Q)_s$, $(P \vee Q)_s$, $(P \rightarrow Q)_s$, or $(P > Q)_s$ where P and Q have $< k$ operators.

($\neg$) A_s is $\neg P_s$. (i) Suppose $v_{w_s}(A) = 1$; then $v_{w_s}(\neg P) = 1$; so by TB($\neg$), $v_{w^*_s}(P) = 0$; so by construction, $v_{w_{\overline{s}}}(P) = 0$; so by assumption, $\Gamma' \nvdash^*_{NBx} P_{\overline{s}}$; so by s-maximality, $\Gamma' \vdash^*_{NBx} \neg P_s$, where this is to say, $\Gamma' \vdash^*_{NBx} A_s$. (ii) Suppose $\Gamma' \vdash^*_{NBx} A_s$; then $\Gamma' \vdash^*_{NBx} \neg P_s$; so by consistency, $\Gamma' \nvdash^*_{NBx} P_{\overline{s}}$; so by assumption, $v_{w_{\overline{s}}}(P) = 0$; so by construction, $v_{w^*_s}(P) = 0$; so by TB($\neg$), $v_{w_s}(\neg P) = 1$, where this is to say, $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

($\wedge$)

($\vee$)

($\rightarrow$) A_s is $(P \rightarrow Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NBx} A_s$; then $v_{w_s}(P \rightarrow Q) = 1$ but $\Gamma' \nvdash^*_{NBx} (P \rightarrow Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NBx} \neg(P \rightarrow Q)_{\overline{s}}$; so, since Γ' is a scapegoat set for $\rightarrow$, there are some y and z such that $\Gamma' \vdash^*_{NBx} s.y.z$, $\Gamma' \vdash^*_{NBx} P_y$ and $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{z}}$; from the latter, by consistency, $\Gamma' \nvdash^*_{NBx} Q_z$; so by assumption, $v_{w_y}(P) = 1$ and $v_{w_z}(Q) = 0$; but since $\Gamma' \vdash^*_{NBx} s.y.z$, by construction, $\langle w_s, w_y, w_z \rangle \in R$; so by TB($\rightarrow$), $v_{w_s}(P \rightarrow Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NBx} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NBx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NBx} (P \rightarrow Q)_s$ but $v_{w_s}(P \rightarrow Q) = 0$. From the latter, by TB($\rightarrow$), there are some $w_t, w_u \in W$ such that $\langle w_s, w_t, w_u \rangle \in R$ and $v_{w_t}(P) = 1$ but $v_{w_u}(Q) = 0$; so by assumption, $\Gamma' \vdash^*_{NBx} P_t$ and $\Gamma' \nvdash^*_{NBx} Q_u$; so by s-maximality, $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{u}}$. Since $\langle w_s, w_t, w_u \rangle \in R$, by construction, $\Gamma' \vdash^*_{NBx} s.t.u$; so by reasoning as follows,

$$
\begin{array}{r|l|ll}
1 & \multicolumn{3}{l}{\Gamma'} \\
2 & & \underline{(P \rightarrow Q)_s} & \text{A } (c, \neg\text{I}) \\
3 & & s.t.u & \text{from } \Gamma' \\
4 & & P_t & \text{from } \Gamma' \\
5 & & Q_u & 3,2,4 \rightarrow\text{E} \\
6 & & \neg Q_{\overline{u}} & \text{from } \Gamma' \\
7 & \multicolumn{2}{l}{\neg(P \rightarrow Q)_{\overline{s}}} & 2\text{-}6\ \neg\text{I}
\end{array}
$$

$\Gamma' \vdash^*_{NBx} \neg(P \rightarrow Q)_{\overline{s}}$; so by consistency, $\Gamma' \nvdash^*_{NBx} (P \rightarrow Q)_s$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NBx} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{NBx} A_s$.

($>$) A_s is $(P > Q)_s$. (i) Suppose $v_{w_s}(A) = 1$ but $\Gamma' \nvdash^*_{NBx} A_s$; then $v_{w_s}(P > Q) = 1$ but $\Gamma' \nvdash^*_{NBx} (P > Q)_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NBx} \neg(P > Q)_{\overline{s}}$; so, since Γ' is a scapegoat set for $>$, there is some y such that $\Gamma' \vdash^*_{NBx} P_{s/y}$, and $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{y}}$; from the first of these, by construction, $\langle w_s, w_y \rangle \in R_P$; and from the second, by consistency, $\Gamma' \nvdash^*_{NBx} Q_y$; so by assumption, $v_{w_y}(Q) = 0$; so by TB($>$), $v_{w_s}(P > Q) = 0$. This is impossible; reject the assumption: if $v_{w_s}(A) = 1$ then $\Gamma' \vdash^*_{NBx} A_s$.

(ii) Suppose $\Gamma' \vdash^*_{NBx} A_s$ but $v_{w_s}(A) = 0$; then $\Gamma' \vdash^*_{NBx} (P > Q)_s$ but $v_{w_s}(P > Q) = 0$. From the latter, by TB($>$), there is a w_t such that $\langle w_s, w_t \rangle \in R_P$, and $v_{w_t}(Q) = 0$; so by assumption, $\Gamma' \nvdash^*_{NBx} Q_t$; so by s-maximality, $\Gamma' \vdash^*_{NBx} \neg Q_{\overline{t}}$. Since $\langle w_s, w_t \rangle \in R_P$, by construction, $\Gamma' \vdash^*_{NBx} P_{s/t}$; so by reasoning as follows,

$$
\begin{array}{r|l|ll}
1 & \multicolumn{3}{l}{\Gamma'} \\
2 & & \underline{(P > Q)_s} & \text{A } (c, \neg\text{I}) \\
3 & & P_{s/t} & \text{from } \Gamma' \\
4 & & Q_t & 2,3 >\text{E} \\
5 & & \neg Q_{\overline{t}} & \text{from } \Gamma' \\
6 & \multicolumn{2}{l}{\neg(P > Q)_{\overline{s}}} & 2\text{-}5\ \neg\text{I}
\end{array}
$$

$\Gamma' \vdash^*_{NBx} \neg(P > Q)_{\overline{s}}$; so by consistency, $\Gamma' \nvdash^*_{NBx} (P > Q)_s$.

This is impossible; reject the assumption: if $\Gamma' \vdash_{\mathit{NBx}} A_s$ then $v_{w_s}(A) = 1$. So $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{\mathit{NBx}} A_s$.

For any A_s, $v_{w_s}(A) = 1$ iff $\Gamma' \vdash^*_{\mathit{NBx}} A_s$.

L8.9 If Γ_0 is consistent, then I_{Bx} constructed as above is a *Bx* interpretation.

In each case, we need to show that relevant constraints are met. Suppose Γ_0 is consistent. By L8.7 Γ' is a scapegoat set for C9/C10 and C12 in those systems.

(NC) Suppose $\langle w_0, w_s, w_t\rangle \in R$; then by construction, $\Gamma' \vdash^*_{\mathit{NBx}} 0.s.t$; so by 0E, $\Gamma' \vdash^*_{\mathit{NBx}} s \simeq t$; so by construction, $w_s = w_t$. Suppose $w_s = w_t$; then by construction, $\Gamma' \vdash^*_{\mathit{NBx}} s \simeq t$; so by 0I, $\Gamma' \vdash^*_{\mathit{NBx}} 0.s.t$; so by construction, $\langle w_0, w_s, w_t\rangle \in R$. So $\langle w_0, w_s, w_t\rangle \in R$ iff $w_s = w_t$; and, since $N = \{w_0\}$, NC is satisfied.

(C8) If C8 is in *Bx*, then AM8 is in *NBx*. Suppose $\langle w_s, w_t, w_u\rangle \in R$; then by construction, $\Gamma' \vdash^*_{\mathit{NBx}} s.t.u$; so by AM8, $\Gamma' \vdash^*_{\mathit{NBx}} s.\overline{u}.\bar{t}$; so by construction, $\langle w_s, w_{\overline{u}}, w_{\bar{t}}\rangle \in R$; so by construction, $\langle w_s, w^*_u, w^*_t\rangle \in R$. So C8 is satisfied.

(C9/10) Suppose there is a w_u such that $\langle w_s, w_t, w_u\rangle \in R$ and $\langle w_u, w_v, w_w\rangle \in R$; then by construction, $\Gamma' \vdash^*_{\mathit{NBx}} s.t.u$ and $\Gamma' \vdash^*_{\mathit{NBx}} u.v.w$; so, since Γ' is a C9/C10 scapegoat set, there is a y such that $\Gamma' \vdash^*_{\mathit{NBx}} s.v.y$ and $\Gamma' \vdash^*_{\mathit{NBx}} t.y.w$, and there is a z such that $\Gamma' \vdash^*_{\mathit{NBx}} t.v.z$ and $\Gamma' \vdash^*_{\mathit{NBx}} s.z.w$; so by construction, $\langle w_s, w_v, w_y\rangle \in R$, $\langle w_t, w_y, w_w\rangle \in R$, $\langle w_t, w_v, w_z\rangle \in R$ and $\langle w_s, w_z, w_w\rangle \in R$. So C9 and C10 are satisfied.

(C12) Similarly.

(C13) If C13 is in *Bx*, then AM13 is in *NBx*. Suppose $\langle w_s, w_t, w_u\rangle \in R$ and $\langle w_u, w_v, w_w\rangle \in R$; then by construction, $\Gamma' \vdash^*_{\mathit{NBx}} s.t.u$ and $\Gamma' \vdash^*_{\mathit{NBx}} u.v.w$; so by AM13, $\Gamma' \vdash^*_{\mathit{NBx}} s.v.w$; so by construction, $\langle w_s, w_v, w_w\rangle \in R$. So C13 is satisfied.

($\preceq$) If ($\preceq$) is in *Bx*, then AM$\preceq$ is in *NBx*. (i) Suppose $\langle w_s, w_t, w_u\rangle \in R$ and $v_{w_s}(p) = 1$; then by construction, $\Gamma' \vdash^*_{\mathit{NBx}} s.t.u$ and $\Gamma' \vdash^*_{\mathit{NBx}} p_s$; so by AM$\preceq$, $\Gamma' \vdash^*_{\mathit{NBx}} p_u$; so by construction, $v_{w_u}(p) = 1$. (ii) Suppose $\langle w_s, w_t, w_u\rangle \in R$ and $v_{w^*_u}(p) = 1$; then by construction, $v_{w_{\overline{u}}}(p) = 1$ so by construction again, $\Gamma' \vdash^*_{\mathit{NBx}} s.t.u$ and $\Gamma' \vdash^*_{\mathit{NBx}} p_{\overline{u}}$; so by AM$\preceq$, $\Gamma' \vdash^*_{\mathit{NBx}} p_{\overline{s}}$; so by construction,

$v_{w_{\bar{s}}}(p) = 1$; and by construction again, $v_{w_s^*}(p) = 1$. So C13 is satisfied.

(1) If condition (1) is in *Bx*, then AMp1 is in *NBx*. Suppose $w_t \in f_A(w_0)$; then $\langle w_0, w_t \rangle \in R_A$; so by construction, $\Gamma' \vdash^*_{NBx} A_{0/t}$; so by AMp1, $\Gamma' \vdash^*_{NBx} A_t$; so by L8.8, $v_{w_t}(A) = 1$; so $w_t \in [A]$. So $f_A(w_0) \subseteq [A]$ and (1) is satisfied.

(2) If condition (2) is in *Bx*, then AMp2 is in *NBx*. Suppose $w_0 \in [A]$; then $v_{w_0}(A) = 1$; so by L8.8, $\Gamma' \vdash^*_{NBx} A_0$; so by AMp2, $\Gamma' \vdash^*_{NBx} A_{0/0}$; so by construction, $\langle w_0, w_0 \rangle \in R_A$; so $w_0 \in f_A(w_0)$ and (2) is satisfied.

Map For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L8.10 If Γ_0 is consistent, then $v_m(\Gamma_0) = 1$.

Reasoning parallel to that for L2.10 of *NK*α.

Main result: Suppose $\Gamma \models_{Bx} A$ but $\Gamma \nvdash_{NBx} A$. Then $\Gamma_0 \models^*_{Bx} A_0$ but $\Gamma_0 \nvdash^*_{NBx} A_0$. By (DN), if $\Gamma_0 \vdash^*_{NBx} \neg\neg A_0$, then $\Gamma_0 \vdash^*_{NBx} A_0$; so $\Gamma_0 \nvdash^*_{NBx} \neg\neg A_0$; so by L8.2, $\Gamma_0 \cup \{\neg A_{\bar{0}}\}$ is consistent; so by L8.9 and L8.10, there is a *Bx* interpretation with v and m constructed as above such that $v_m(\Gamma_0 \cup \{\neg A_{\bar{0}}\}) = 1$; so $v_{m(\bar{0})}(\neg A) = 1$; so by construction, $v_{m_0^*}(\neg A) = 1$; so by TB($\neg$), $v_{m(0)}(A) = 0$; so $v_m(\Gamma_0) = 1$ and $v_{m(0)}(A) = 0$; so by VBx*, $\Gamma_0 \not\models^*_{Bx} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{Bx} A$, then $\Gamma \vdash_{NBx} A$.

9 Four-Valued Relevant Logics: *R4x* (ch. 10,11)

Though Priest does not do so — and it has been suggested that it cannot reasonably be done [4], relevant systems are capable of a four-valued treatment. Thus, to make contact with what has gone before, and contact with some of my own suggestions for the significance of relevant semantics [5], a four-valued approach is developed. The discussion is restricted to (standard) logics in the range DW - R — though it might be extended beyond.

9.1 Language / Semantic Notions

LR4 The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators $\neg$, $\wedge$, $\vee$, and $\rightarrow$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \rightarrow B)$. $A \supset B$ abbreviates $\neg A \vee B$. In the extended case, the language includes $\Box$; then if A is a formula, so is $\Box A$; and $\Diamond A$ abbreviates $\neg\Box\neg A$. If A is a formula so formed, so is $\overline{A}$.

Let $/A/$ and $\backslash A\backslash$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $/A/$ and $\backslash A\backslash$ are opposite. And similarly for other expressions with overlines as below.

IR4 Without $\Box$ in the language, an INTERPRETATION is $\langle W, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ where W is a set of worlds; $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively; $R, \overline{R} \subseteq W^3$ are access relations for truth and non-falsity respectively; and h is a valuation which assigns to each $/p/$ either 1 or 0 at each $w \in W$. $\preceq$ encompasses the inclusion relations $\leq$, $\leq^*$ and $\leq^\sharp$, constrained so that,

($\preceq$) Each of the following obtain,

$$a \leq b \Rightarrow \begin{cases} \text{if } h_a(p) = 1 \text{ then } h_b(p) = 1 \text{ and if } h_b(\overline{p}) = 1 \text{ then } h_a(\overline{p}) = 1 \\ \text{if } bRxy \text{ then } aRxy \text{ if } a \notin N\text{, otherwise if } bRxy \text{ then } x \leq y \\ \text{if } a\overline{R}xy \text{ then } b\overline{R}xy \text{ if } b \notin \overline{N}\text{, otherwise if } a\overline{R}xy \text{ then } x \leq y \end{cases}$$

$$a \leq^* b \Rightarrow \begin{cases} \text{if } h_a(p) = 1 \text{ then } h_b(\overline{p}) = 1 \text{ and if } h_b(p) = 1 \text{ then } h_a(\overline{p}) = 1 \\ \text{if } b\overline{R}xy \text{ then } aRxy \text{ if } a \notin N\text{, otherwise if } b\overline{R}xy \text{ then } x \leq y \\ \text{if } a\overline{R}xy \text{ then } bRxy \text{ if } b \notin N\text{, otherwise if } a\overline{R}xy \text{ then } x \leq y \end{cases}$$

$$a \leq^\sharp b \Rightarrow \begin{cases} \text{if } h_a(\overline{p}) = 1 \text{ then } h_b(p) = 1 \text{ and if } h_b(\overline{p}) = 1 \text{ then } h_a(p) = 1 \\ \text{if } bRxy \text{ then } a\overline{R}xy \text{ if } a \notin \overline{N}\text{, otherwise if } bRxy \text{ then } x \leq y \\ \text{if } aRxy \text{ then } b\overline{R}xy \text{ if } b \notin \overline{N}\text{, otherwise if } aRxy \text{ then } x \leq y \end{cases}$$

As additional constraints on interpretations, we may require any of, NC For any $w \in /N/$, $w/R/xy$ iff $x = y$

C^9_{10} If $a/R/bx$ and $xRcd$ then there is a y such that $bRcy$ and $a/R/yd$, and a z such that $bRzd$ and $a/R/cz$. And if $a/R/xb$ and $x\overline{R}cd$ then there is a y such that $b\overline{R}cy$ and $a/R/yd$, and a z such that $b\overline{R}zd$ and $a/R/cz$.

C11 If $a/R/bc$ then there is a y such that $a/R/by$ and $yRbc$ and a z such that $a/R/zc$ and $z\overline{R}bc$.

C12 If $aRbc$ then for some $y \geq a$, $bRyc$, and for some $z \geq^* a$, $c\overline{R}bz$. And if $a\overline{R}bc$ then for some $y \geq^\sharp a$, $bRyc$, and for some $z \leq a$, $c\overline{R}bz$

CL (i) $w \in N$ iff $w \in \overline{N}$

(ii) for any $w \in N$, $h_w(\overline{p}) = h_w(p)$.

Then the base standard relevant system is DW and includes just NC. Other regular relevant systems add from C9 - C12 in the usual way [8]; so TW has C^9_{10}, RW adds C12, and R all three. The 4A systems from [5] drop NC (and, for that matter N, $\overline{N}$ and M) but may include C9 - C12; the 4B systems from [5] include NC, and might include any of the other constraints, including CL.

Where the language includes $\Box$, 4B interpretations may be extended to be of the sort, $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ where $M \subseteq W$ is a modal access relation. Interpretations are subject to,

MC If $w \in /N/$ and wMx, then $x \in /N/$

CM Where $\preceq$ is any of the three inclusion relations, require: If $a \preceq b$ then (i) if bMc there is some $y \preceq c$ such that aMy; and (ii) if aMc then there is some $y \succeq c$ such that bMy.

and optionally standard modal constraints of the sort,

κ If $a/R/bx$ and xMc then there is a y such that bMy and $a/R/yc$, and if $a/R/xb$ and xMc, there is a y such that bMy and $a/R/cy$.

ρ Reflexivity: for all x, xMx.

σ Symmetry: for all x, y if xMy then yMx.

τ Transitivity: for all x, y, z if xMy and yMz then xMz.

The last three give results we expect as T: $\Box A \rightarrow A$; B: $A \rightarrow \Box\Diamond A$ and 4: $\Box A \rightarrow \Box\Box A$. Without constraint, $A \rightarrow B \models_{NR4m} \Box A \rightarrow \Box B$; $(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ comes from (κ); the K principle $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ then comes together with reflexivity. It is possible to obtain the K principle independently, but the required constraint with conditions like, (κ) if $a/R/bx$ and xMc, then there are y, z such that aMy, bMz and $y/R/zc$, is relatively difficult to motivate. For discussion see [5] note 14.

In the absence of C12 it is simplest to omit inclusion relations (or set them to the empty set); and similarly in the absence of modal operators to omit modal access (or to set it to the empty set).

HR4 For complex expressions,

- $(\neg)$ $h_w(/\neg P/) = 1$ iff $h_w(\backslash P\backslash) = 0$
- $(\wedge)$ $h_w(/P \wedge Q/) = 1$ iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$
- $(\vee)$ $h_w(/P \vee Q/) = 1$ iff $h_w(/P/) = 1$ or $h_w(/Q/) = 1$
- $(\rightarrow)$ $h_w(/P \rightarrow Q/) = 1$ iff there are no $x, y \in W$ such that $w/R/xy$ and $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$
- $(\Box)$ $h_w(/\Box P/) = 1$ iff there is no $x \in W$ such that wMx and $h_x(/P/) = 0$

For a set Γ of formulas, $h_w(\Gamma) = 1$ iff $h_w(/P/) = 1$ for each $/P/ \in \Gamma$; then,

VR4 $\Gamma \models_{R4x} P$ iff there is no R4x interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

9.2 Natural Derivations: *NR4x*

Allow subscripts and expressions of the sort $s.t$, $/r.s.t/$, $s \simeq t$, and $s \preceq t$ (for each of the three inclusions). Allow also $/n/[s]$ and $\neg/n/[s]$ (with implicit subscript s); to say that a world is or is not in $/N/$; where P_s is $/n/[s]$ or $\neg/n/[s]$, let $/P/_s$ and $\backslash P\backslash_s$ be the same expression, so that $/n/[s]$ and $\neg/n/[s]$ contradict for ¬I.

$$\mathbf{R}\ \begin{array}{|l} /P/_s \\ \\ /P/_s \end{array} \qquad \neg\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} /P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ \backslash\neg P\backslash_s \end{array} \qquad \neg\mathbf{E}\ \begin{array}{|l} \begin{array}{|l} /\neg P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ \backslash P\backslash_s \end{array}$$

$$\begin{array}{r|l} \wedge\mathbf{I} & /P/_s \\ & /Q/_s \\ & \\ & /P \wedge Q/_s \end{array} \qquad \begin{array}{r|l} \wedge\mathbf{E} & /P \wedge Q/_s \\ & \\ & /P/_s \end{array} \qquad \begin{array}{r|l} \wedge\mathbf{E} & /P \wedge Q/_s \\ & \\ & /Q/_s \end{array}$$

$$\begin{array}{r|l} \vee\mathbf{I} & /P/_s \\ & \\ & /P \vee Q/_s \end{array} \qquad \begin{array}{r|l} \vee\mathbf{I} & /P/_s \\ & \\ & /Q \vee P/_s \end{array} \qquad \begin{array}{r|l} \vee\mathbf{E} & /P \vee Q/_s \\ & \begin{array}{|l} \underline{/P/_s} \\ \\ /\!/R/\!/_t \end{array} \\ & \begin{array}{|l} \underline{/Q/_s} \\ \\ /\!/R/\!/_t \end{array} \\ & /\!/R/\!/_t \end{array}$$

$$\begin{array}{r|l} \supset\mathbf{I} & \begin{array}{|l} \underline{/P/_s} \\ \\ \backslash Q\backslash_s \end{array} \\ & \backslash P \supset Q\backslash_s \end{array} \qquad \begin{array}{r|l} \supset\mathbf{E} & \backslash P \supset Q\backslash_s \\ & /P/_s \\ & \\ & \backslash Q\backslash_s \end{array}$$

$$\begin{array}{r|l|l} \rightarrow\mathbf{E} & /s.t.u/ & /s.t.u/ \\ & /P \rightarrow Q/_s & /P \rightarrow Q/_s \\ & P_t & \overline{P}_u \\ & & \\ & Q_u & \overline{Q}_t \end{array} \qquad \begin{array}{r|l|l} \rightarrow\mathbf{I} & \begin{array}{|l} /s.t.u/ \\ \underline{P_t} \\ \\ Q_u \end{array} & \begin{array}{|l} /s.t.u/ \\ \underline{\overline{P}_u} \\ \\ \overline{Q}_t \end{array} \\ & /P \rightarrow Q/_s & /P \rightarrow Q/_s \end{array} \qquad \begin{array}{r|l|l} \mathbf{CL} & /n/[s] & /n/[s] \\ & /\!/P/\!/_s & \\ & & \\ & \backslash\!\backslash P\backslash\!\backslash_s & \backslash n\backslash[s] \end{array}$$

where t and u do not appear in any undischarged premise or assumption

$$\begin{array}{r|l} \mathbf{NI} & \\ & \\ & n[0] \end{array} \qquad \begin{array}{r|l|l} \mathbf{NE} & /n/[a] & /n/[a] \\ & s \simeq t & /a.s.t/ \\ & & \\ & /a.s.t/ & s \simeq t \end{array} \qquad \begin{array}{r|l} \simeq\mathbf{I} & \\ & \\ & s \simeq s \end{array} \qquad \begin{array}{r|l} \simeq\mathbf{E} & s \simeq t \\ & \mathcal{P}(s) \\ & \\ & \mathcal{P}(t) \end{array}$$

These are the rules for the base systems. DW takes all the rules but CL. Roy's 4A drops the NI, NE, ≃I and ≃E rules. Roy's 4B is like DW except that it may include CL. From these it is possible to add from the following in the natural way.

$$\begin{array}{r|l} \Box\mathbf{I} & \begin{array}{|l} \underline{s.t} \\ \\ /P/_t \end{array} \\ & /\Box P/_s \end{array} \qquad \begin{array}{r|l} \Box\mathbf{E} & /\Box P/_s \\ & s.t \\ & \\ & /P/_t \end{array} \qquad \begin{array}{r|l} \Diamond\mathbf{I} & /P/_t \\ & s.t \\ & \\ & /\Diamond P/_s \end{array} \qquad \begin{array}{r|l} \Diamond\mathbf{E} & /\Diamond P/_s \\ & \begin{array}{|l} s.t \\ \underline{/P/_t} \\ \\ /\!/Q/\!/_u \end{array} \\ & /\!/Q/\!/_u \end{array} \qquad \begin{array}{r|l} \mathbf{MC} & /n/[s] \\ & s.t \\ & \\ & /n/[t] \end{array}$$

where t does not appear in any undischarged premise or assumption

where t does not appear in any undischarged premise or assumption and is not u

$$\mathbf{AM}\kappa\ \begin{array}{|l} /a.b.x/ \\ x.c \\ \begin{array}{|l} /a.y.c/ \\ \underline{b.y} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} /a.x.b/ \\ x.c \\ \begin{array}{|l} /a.c.y/ \\ \underline{b.y} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \mathbf{AM}\rho\ \begin{array}{|l} \\ \\ \\ s.s \end{array} \qquad \mathbf{AM}\sigma\ \begin{array}{|l} s.t \\ \\ \\ t.s \end{array} \qquad \mathbf{AM}\tau\ \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

where y does not appear in an undischarged premise or assumption and is not w

$$\mathbf{AM}^9_{10}\ \begin{array}{|l} /a.b.x/ \\ x.c.d \\ \begin{array}{|l} b.c.y \\ \underline{/a.y.d/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} /a.b.x/ \\ x.c.d \\ \begin{array}{|l} b.y.d \\ \underline{/a.c.y/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} /a.x.b/ \\ \overline{x.c.d} \\ \begin{array}{|l} \overline{b.c.y} \\ \underline{/a.y.d/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} /a.x.b/ \\ \overline{x.c.d} \\ \begin{array}{|l} \overline{b.y.d} \\ \underline{/a.c.y/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array}$$

$$\mathbf{AM11}\ \begin{array}{|l} /s.t.u/ \\ \begin{array}{|l} /s.t.y/ \\ \underline{y.t.u} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} /s.t.u/ \\ \begin{array}{|l} /s.y.u/ \\ \underline{\overline{y.t.u}} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \leq\mathbf{E}\ \begin{array}{|l} a \leq b \\ P_a \\ \\ P_b \end{array} \qquad \leq^*\mathbf{E}\ \begin{array}{|l} a \leq^* b \\ P_a \\ \\ \overline{P}_b \end{array} \qquad \leq^\sharp\mathbf{E}\ \begin{array}{|l} a \leq^\sharp b \\ \overline{P}_a \\ \\ P_b \end{array}$$

$$\mathbf{AM12}\ \begin{array}{|l} a.b.c \\ \begin{array}{|l} y \geq a \\ \underline{b.y.c} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} a.b.c \\ \begin{array}{|l} y \geq^* a \\ \underline{\overline{c.b.y}} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} \overline{a.b.c} \\ \begin{array}{|l} y \geq^\sharp a \\ \underline{b.y.c} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \qquad \begin{array}{|l} \overline{a.b.c} \\ \begin{array}{|l} y \leq a \\ \underline{\overline{c.b.y}} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array}$$

where y does not appear in any undischarged premise or assumption and is not w

◊I and ◊E are derived. Though they will not play a natural role in most derivations, for systems with the inclusion relations (and so for systems like R with AM12) we also allow also:

$\leq_R$E: $a \leq b,\ b.x.y,\ n[a] \vdash x \leq y$; $a \leq b,\ b.x.y,\ {\sim}n[a] \vdash a.x.y$

$a \leq b,\ \overline{a.x.y},\ \overline{n}[b] \vdash x \leq y$; $a \leq b,\ \overline{a.x.y},\ {\sim}\overline{n}[b] \vdash \overline{b.x.y}$

$\leq^*_R$E: $a \leq^* b,\ \overline{b.x.y},\ n[a] \vdash x \leq y$; $a \leq^* b,\ \overline{b.x.y},\ {\sim}n[a] \vdash a.x.y$

$a \leq^* b,\ \overline{a.x.y},\ n[b] \vdash x \leq y$; $a \leq^* b,\ \overline{a.x.y},\ {\sim}n[b] \vdash b.x.y$

$\leq^\sharp_R$E: $a \leq^\sharp b,\ b.x.y,\ \overline{n}[a] \vdash x \leq y;\ a \leq^\sharp b,\ b.x.y,\ {\sim}\overline{n}[a] \vdash \overline{a.x.y}$

$a \leq^\sharp b,\ a.x.y,\ \overline{n}[b] \vdash x \leq y;\ a \leq^\sharp b,\ a.x.y,\ {\sim}\overline{n}[b] \vdash \overline{b.x.y}$

$$\preceq_M\text{E}\ \begin{array}{|l} a \preceq b \\ b.c \\ \begin{array}{|l} y \preceq c \\ \underline{a.y} \\ \\ \\ /P/_w \end{array} \\ /P/_w \end{array} \qquad\qquad \preceq_M\text{E}\ \begin{array}{|l} a \preceq b \\ a.c \\ \begin{array}{|l} c \preceq y \\ \underline{b.y} \\ \\ \\ /P/_w \end{array} \\ /P/_w \end{array}$$

where y does not appear in any undischarged premise or assumption and is not w

Every subscript is 0 or is introduced according to the rules in an assumption. Where Γ is a set of unsubscripted formulas, let Γ_0 be those same formulas, each with subscript 0. Then,

VNR4 $\Gamma \vdash_{NR4x} A$ iff there is an *NR4x* derivation of A_0 from the members of Γ_0.

Examples. Here are some cases, with the first ones paired to illustrate the match between derivations that do, and ones that do not, include the NI, NE, $\simeq$I and $\simeq$E rules.

$(A \to B) \wedge (A \to C) \vdash_{NR4Ax} A \to (B \wedge C)$

1	$\vert\ \underline{(A \to B) \wedge (A \to C)_0}$	P
2	$\vert\ \vert\ 0.1.2$	A (g, →I)
3	$\vert\ \vert\ \underline{A_1}$	
4	$\vert\ \vert\ (A \to B)_0$	1 ∧E
5	$\vert\ \vert\ (A \to C)_0$	1 ∧E
6	$\vert\ \vert\ B_2$	2,4,3 →E
7	$\vert\ \vert\ C_2$	2,5,3 →E
8	$\vert\ \vert\ (B \wedge C)_2$	6,7 ∧I
9	$\vert\ A \to (B \wedge C)_0$	2-8 →I

(A5) $\vdash_{NR4Bx} [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

1	$\mid\ \mid\ 0.1.2$	A (g, →I)
2	$\mid\ \mid\ \underline{(A \to B) \wedge (A \to C)_1}$	
3	$\mid\ \mid\ n[0]$	NI
4	$\mid\ \mid\ 1 \simeq 2$	1,3 NE
5	$\mid\ \mid\ \mid\ 2.3.4$	A (g, →I)
6	$\mid\ \mid\ \mid\ \underline{A_3}$	
7	$\mid\ \mid\ \mid\ (A \to B) \wedge (A \to C)_2$	2,4 $\simeq$E
8	$\mid\ \mid\ \mid\ (A \to B)_2$	7 ∧E
9	$\mid\ \mid\ \mid\ B_4$	5,8,6 →E
10	$\mid\ \mid\ \mid\ (A \to C)_2$	7 ∧E
11	$\mid\ \mid\ \mid\ C_4$	5,10,6 →E
12	$\mid\ \mid\ \mid\ (B \wedge C)_4$	9,11 ∧I
13	$\mid\ \mid\ [A \to (B \wedge C)]_2$	5-12 →I
14	$\mid\ [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]_0$	1-13 →I

$A \to \neg B \vdash_{NR4Ax} B \to \neg A$

1	$\mid\ \underline{(A \to \neg B)_0}$	P
2	$\mid\ \mid\ 0.1.2$	A (g, →I)
3	$\mid\ \mid\ \underline{B_1}$	
4	$\mid\ \mid\ \mid\ \underline{\overline{A}_2}$	A (c, ¬I)
5	$\mid\ \mid\ \mid\ \overline{\neg B}_1$	2,1,4 →E
6	$\mid\ \mid\ \mid\ B_1$	3 R
7	$\mid\ \mid\ \neg A_2$	4-6 ¬I
8	$\mid\ (B \to \neg A)_0$	2-7 →I

(A8) $\vdash_{NR4Bx} (A \to \neg B) \to (B \to \neg A)$

1	$0.1.2$	A (g, $\to$I)
2	$(A \to \neg B)_1$	
3	$n[0]$	NI
4	$1 \simeq 2$	3,1 NE
5	$2.3.4$	A (g, $\to$I)
6	B_3	
7	$\overline{A}_4$	A (c, $\neg$I)
8	$(A \to \neg B)_2$	2,4 $\simeq$E
9	$\overline{\neg B}_3$	5,8,7 $\to$E
10	B_3	6 R
11	$\neg A_4$	7-10 $\neg$I
12	$(B \to \neg A)_2$	5-11 $\to$I
13	$(A \to \neg B) \to (B \to \neg A)_0$	1-12 $\to$I

$\vdash_{NR4r} (\neg A \to A) \to A$

1	$0.1.2$	A (g, $\to$I)
2	$(\neg A \to A)_1$	
3	$n[0]$	NI
4	$0.3.2$	A (g, 1 AM11)
5	$\overline{3.1.2}$	
6	$3 \simeq 2$	3,4 NE
7	$4 \geq^\sharp 3$	A (g, 5 AM12)
8	$1.4.2$	
9	$\overline{\neg A}_2$	A (c, $\neg$E)
10	$\overline{\neg A}_3$	9,6 $\simeq$E
11	$\neg A_4$	7,10 $\leq^\sharp$E
12	A_2	8,2,11 $\to$E
13	$\overline{\neg A}_2$	9 R
14	A_2	9-13 $\neg$E
15	A_2	5,7-14 AM12
16	A_2	1,4-15 AM11
17	$[(\neg A \to A) \to A]_0$	1-16 $\to$I

$A \to B \vdash_{NR4m} \Box A \to \Box B$

1	$(A \to B)_0$	P
2	$0.1.2$	A (g, $\to$I)
3	$\Box A_1$	
4	$n[0]$	NI
5	$1 \simeq 2$	2,4 NE
6	2.3	A (g, $\Box$I)
7	$\Box A_2$	3,5 $\simeq$E
8	A_3	6,7 $\Box$E
9	$3 \simeq 3$	$\simeq$I
10	$0.3.3$	4,9 NE
11	B_3	10,1,8 $\to$E
12	$\Box B_2$	6-11 $\Box$I
13	$(\Box A \to \Box B)_0$	2-12 $\to$I

9.3 Soundness and Completeness: *R4x*

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$, let m be a map from subscripts into W such that $m(0)$ is some member of N. Then say $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ is $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ *with* map m. Let $h_{m(s)}[/n/(s)] = 1$ iff $m(s) \in /N/$ and $h_{m(s)}[\neg/n/(s)] = 1$ iff $h_{m(s)}[/n/(s)] = 0$. Then, where Γ is a set of expressions of our language for derivations, $h_m(\Gamma) = 1$ iff for each $/A_s/ \in \Gamma$, $h_{m(s)}(/A/) = 1$, for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in M$, for each $/r.s.t/ \in \Gamma$, $\langle m(r), m(s), m(t)\rangle \in /R/$, for each $s \simeq t$ in Γ, $m(s) = m(t)$ and for each $s \preceq t$ in Γ, $\langle m(s), m(t)\rangle \in \preceq$ (for each of $\leq$, $\leq^*$ and $\leq^\sharp$ in $\preceq$). Now expand notions of validity for subscripts, overlines, and alternate expressions as indicated in double brackets as follows,

VR4x* $\Gamma \vDash^*_{R4x} /A/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t\rrbracket$ iff there is no *R4x* interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma) = 1$ but $h_{m(s)}(/A/) = 0$ $\llbracket\langle m(s),\ m(t)\rangle \notin M,\ \langle m(r), m(s), m(t)\rangle \notin /R/,\ m(s) \neq m(t),\ \langle m(s),\ m(t)\rangle \notin \preceq\rrbracket$.

NR4x* $\Gamma \vdash^*_{NR4x} /A/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t\rrbracket$ iff there is an *NR4x* derivation of $/A/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t\rrbracket$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A are without overlines and have subscript 0 (and so do not include expressions of the sort $s.t$, $/r.s.t/$, or $s \preceq t$). As usual, for the following, cases omitted are like ones worked, and so left to the reader.

THEOREM 9.1 *NR4x is sound: If* $\Gamma \vdash_{NR4x} A$ *then* $\Gamma \models_{R4x} A$.

L9.0 If $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ is an $R4x$ interpretation, then there is an $R4x$ interpretation $\langle \mathsf{W}, \mathsf{M}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \preccurlyeq, \mathsf{h}\rangle$ with $\mathsf{w}, \mathsf{w}^* \in \mathsf{W}$ corresponding to each $w \in W$ such that for any $/A/$, (i) $\mathsf{h}_\mathsf{w}(/A/) = 1$ iff $h_w(/A/) = 1$ and (ii) $\mathsf{h}_{\mathsf{w}^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$.

For $R4x$ interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$, consider $\langle \mathsf{W}, \mathsf{M}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \preccurlyeq, \mathsf{h}\rangle$ such that corresponding to each $w \in W$ there are $\mathsf{w}, \mathsf{w}^* \in \mathsf{W}$ such that, $\mathsf{M} = \{\langle \mathsf{w}, \mathsf{x}\rangle, \langle \mathsf{w}^*, \mathsf{x}^*\rangle \mid \langle w, x\rangle \in M\}$; $\mathsf{w} \in /\mathsf{N}/$ iff $w \in /N/$, and $\mathsf{w}^* \in /\mathsf{N}/$ iff $w \in \backslash N\backslash$. Set,

$/\mathsf{R}/ = \{\langle \mathsf{x}, \mathsf{y}, \mathsf{z}\rangle, \langle \mathsf{x}, \mathsf{z}^*, \mathsf{y}^*\rangle \mid \langle x/R/yz\} \cup \{\langle \mathsf{x}^*, \mathsf{y}, \mathsf{z}\rangle, \langle \mathsf{x}^*, \mathsf{z}^*, \mathsf{y}^*\rangle \mid \langle x\backslash R\backslash yz\}$

$\preccurlyeq\, = \{\langle \mathsf{y}, \mathsf{z}\rangle \mid y \leq z\} \cup \{\langle \mathsf{y}^*, \mathsf{z}^*\rangle \mid z \leq y\} \cup \{\langle \mathsf{y}, \mathsf{z}^*\rangle \mid y \leq^* z\} \cup \{\langle \mathsf{y}^*, \mathsf{z}\rangle \mid y \leq^\sharp z\}$

$\preccurlyeq^* = \{\langle \mathsf{y}, \mathsf{z}\rangle \mid y \leq^* z\} \cup \{\langle \mathsf{y}^*, \mathsf{z}^*\rangle \mid y \leq^\sharp z\} \cup \{\langle \mathsf{y}, \mathsf{z}^*\rangle \mid y \leq z\} \cup \{\langle \mathsf{y}^*, \mathsf{z}\rangle \mid z \leq y\}$

$\preccurlyeq^\sharp = \{\langle \mathsf{y}, \mathsf{z}\rangle \mid y \leq^\sharp z\} \cup \{\langle \mathsf{y}^*, \mathsf{z}^*\rangle \mid y \leq^* z\} \cup \{\langle \mathsf{y}, \mathsf{z}^*\rangle \mid z \leq y\} \cup \{\langle \mathsf{y}^*, \mathsf{z}\rangle \mid y \leq z\}$.

And $\mathsf{h}_\mathsf{w}(/p/) = h_w(/p/)$; but $\mathsf{h}_{\mathsf{w}^*}(/p/) = h_w(\backslash p\backslash)$.

(1) By induction on the length of A, for any w, (i) $\mathsf{h}_\mathsf{w}(/A/) = h_w(/A/)$ and (ii) $\mathsf{h}_{\mathsf{w}^*}(/A/) = h_w(\backslash A\backslash)$.

Basis: If A has no operator symbols then A is a parameter p. But then by construction, $\mathsf{h}_\mathsf{w}(/p/) = h_w(/p/)$ and $\mathsf{h}_{\mathsf{w}^*}(/p/) = h_w(\backslash p\backslash)$.

Assp: For any i, $0 \leq i < k$ if A has i operator symbols, then for any w, both (i) and (ii) are met.

Show: If A has k operator symbols, (i) and (ii) are met. If A has k operator symbols, it is of the form $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$ or $\Box P$ where P and Q have $< k$ operator symbols.

($\neg$) $/A/$ is $/\neg P/$. (i) By HR4($\neg$), $\mathsf{h}_\mathsf{w}(/\neg P/) = 1$ iff $\mathsf{h}_\mathsf{w}(\backslash P\backslash) = 0$; by assumption iff $h_w(\backslash P\backslash) = 0$; by HR4($\neg$), iff $h_w(/\neg P/) = 1$. (ii) By HR4($\neg$), $\mathsf{h}_{\mathsf{w}^*}(/\neg P/) = 1$ iff $\mathsf{h}_{\mathsf{w}^*}(\backslash P\backslash) = 0$; by assumption iff $h_w(/P/) = 0$; by HR4($\neg$), iff $h_w(\backslash\neg P\backslash) = 1$.

($\wedge$) $/A/$ is $/P \wedge Q/$. (i) By HR4($\wedge$), $\mathsf{h}_\mathsf{w}(/P \wedge Q/) = 1$ iff $\mathsf{h}_\mathsf{w}(/P/) = 1$ and $\mathsf{h}_\mathsf{w}(/Q/) = 1$; by assumption, iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$; by HR4($\wedge$), iff $h_w(/P \wedge Q/) = 1$. (ii) By HR4($\wedge$), $\mathsf{h}_{\mathsf{w}^*}(/P \wedge Q/) = 1$ iff $\mathsf{h}_{\mathsf{w}^*}(/P/) = 1$ and $\mathsf{h}_{\mathsf{w}^*}(/Q/) = 1$; by assumption, iff $h_w(\backslash P\backslash) = 1$ and $h_w(\backslash Q\backslash) = 1$; by HR4($\wedge$), iff $h_w(\backslash P \wedge Q\backslash) = 1$.

($\vee$)

($\Box$) $/A/$ is $/\Box P/$. (i) By HR4($\Box$) and the construction, $\mathsf{h}_{\mathsf{w}}(/\Box P/) = 0$ iff there is some $\mathsf{x} \in \mathsf{W}$ such that wMx and $\mathsf{h}_{\mathsf{x}}(/P/) = 0$; but by construction wMx iff wMx, and by assumption $\mathsf{h}_{\mathsf{x}}(/P/) = 0$ iff $h_x(/P/) = 0$; by HR4($\Box$), iff $h_w(/\Box P/) = 0$. (ii) By HR4($\Box$) and the construction, $\mathsf{h}_{\mathsf{w}^*}(/\Box P/) = 0$ iff there is some $\mathsf{x}^* \in \mathsf{W}$ such that $\mathsf{w}^*\mathsf{M}\mathsf{x}^*$ and $\mathsf{h}_{\mathsf{x}^*}(/P/) = 0$; but by construction $\mathsf{w}^*\mathsf{M}\mathsf{x}^*$ iff wMx, and by assumption $\mathsf{h}_{\mathsf{x}^*}(/P/) = 0$ iff $h_x(\backslash P\backslash) = 0$; by HR4($\Box$), iff $h_w(\backslash\Box P\backslash) = 0$.

($\rightarrow$) (a) By HR4($\rightarrow$) and construction, $\mathsf{h}_{\mathsf{w}}(/A \rightarrow B/) = 0$ iff there are either $\mathsf{y}, \mathsf{z} \in \mathsf{W}$ such that $\mathsf{w/R/yz}$ and $\mathsf{h}_{\mathsf{y}}(A) = 1$ but $\mathsf{h}_{\mathsf{z}}(B) = 0$ or $\mathsf{h}_{\mathsf{z}}(\overline{A}) = 1$ but $\mathsf{h}_{\mathsf{y}}(\overline{B}) = 0$, or there are $\mathsf{z}^*, \mathsf{y}^* \in \mathsf{W}$ such that $\mathsf{w/R/z^*y^*}$ and $\mathsf{h}_{\mathsf{z}^*}(A) = 1$ but $\mathsf{h}_{\mathsf{y}^*}(B) = 0$ or $\mathsf{h}_{\mathsf{y}^*}(\overline{A}) = 1$ but $\mathsf{h}_{\mathsf{z}^*}(\overline{B}) = 0$. In the first case, by construction $w/R/yz$ and by assumption $h_y(A) = 1$ but $h_z(B) = 0$ or $h_z(\overline{A}) = 1$ but $h_y(\overline{B}) = 0$. In the second case, by construction, $w/R/yz$ and by assumption $h_z(\overline{A}) = 1$ but $h_y(\overline{B}) = 0$ or $h_y(A) = 1$ but $h_z(B) = 0$. By HR4($\rightarrow$) either is so iff $h_w(/A \rightarrow B/) = 0$. (b) By HR4($\rightarrow$) and construction, $\mathsf{h}_{\mathsf{w}^*}(/A \rightarrow B/) = 0$ iff there are either $\mathsf{y}, \mathsf{z} \in \mathsf{W}$ such that $\mathsf{w^*/R/yz}$ and $\mathsf{h}_{\mathsf{y}}(A) = 1$ but $\mathsf{h}_{\mathsf{z}}(B) = 0$ or $\mathsf{h}_{\mathsf{z}}(\overline{A}) = 1$ but $\mathsf{h}_{\mathsf{y}}(\overline{B}) = 0$, or there are $\mathsf{z}^*, \mathsf{y}^* \in \mathsf{W}$ such that $\mathsf{w^*/R/z^*y^*}$ and $\mathsf{h}_{\mathsf{z}^*}(A) = 1$ but $\mathsf{h}_{\mathsf{y}^*}(B) = 0$ or $\mathsf{h}_{\mathsf{y}^*}(\overline{A}) = 1$ but $\mathsf{h}_{\mathsf{z}^*}(\overline{B}) = 0$. In the first case, by construction $w\backslash R\backslash yz$ and by assumption $h_y(A) = 1$ but $h_z(B) = 0$ or $h_z(\overline{A}) = 1$ but $h_y(\overline{B}) = 0$. In the second case, by construction, $w\backslash R\backslash yz$ and by assumption $h_z(\overline{A}) = 1$ but $h_y(\overline{B}) = 0$ or $h_y(A) = 1$ but $h_z(B) = 0$. By HR4($\rightarrow$) either is so iff $h_{w^*}(\backslash A \rightarrow B\backslash) = 0$.

For any w and A, (i) $\mathsf{h}_{\mathsf{w}}(/A/) = h_w(/A/)$ and (ii) $\mathsf{h}_{\mathsf{w}^*}(/A/) = h_w(\backslash A\backslash)$.

(2) If $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ is an *R4x* interpretation then $\langle \mathsf{W}, \mathsf{M}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \preccurlyeq, \mathsf{h}\rangle$ is an *R4x* interpretation.

($\leqslant$) (i) Suppose $\mathsf{a} \leqslant \mathsf{b}$; then by construction $a \leq b$. (a) Suppose $\mathsf{h}_{\mathsf{a}}(p) = 1$; then by (1), $h_a(p) = 1$; so by ($\leq$), $h_b(p) = 1$; so by (1), $\mathsf{h}_{\mathsf{b}}(p) = 1$. Suppose $\mathsf{h}_{\mathsf{b}}(\overline{p}) = 1$; then by (1), $h_b(\overline{p}) = 1$; so by ($\leq$), $h_a(\overline{p}) = 1$; so by (1), $\mathsf{h}_{\mathsf{a}}(\overline{p}) = 1$. (b.i) Suppose bRxy and $\mathsf{a} \notin \mathsf{N}$; then by construction, $bRxy$ and $a \notin N$; so by ($\leq$), $aRxy$ and by construction, aRxy. Suppose bRxy

and $\mathsf{a} \in \mathsf{N}$; then by construction, $bRxy$ and $a \in N$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (b.ii) Suppose $\mathsf{bRy}^*\mathsf{x}^*$ and $\mathsf{a} \notin \mathsf{N}$; then by construction, $bRxy$ and $a \notin N$; so by $(\leq)$, $aRxy$ and by construction, $\mathsf{aRy}^*\mathsf{x}^*$. Suppose $\mathsf{bRy}^*\mathsf{x}^*$ and $\mathsf{a} \in \mathsf{N}$; then by construction, $bRxy$ and $a \in N$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{y}^* \leqslant \mathsf{x}^*$. (c.i) Suppose $\mathsf{a}\overline{\mathsf{R}}\mathsf{xy}$ and $\mathsf{b} \notin \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \notin \overline{N}$; so by $(\leq)$, $b\overline{R}xy$ and by construction, $\mathsf{b}\overline{\mathsf{R}}\mathsf{xy}$. Suppose $\mathsf{a}\overline{\mathsf{R}}\mathsf{xy}$ and $\mathsf{b} \in \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \in \overline{N}$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (c.ii) Suppose $\mathsf{a}\overline{\mathsf{R}}\mathsf{y}^*\mathsf{x}^*$ and $\mathsf{b} \notin \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \notin \overline{N}$; so by $(\leq)$, $b\overline{R}xy$ and by construction, $\mathsf{b}\overline{\mathsf{R}}\mathsf{y}^*\mathsf{x}^*$. Suppose $\mathsf{a}\overline{\mathsf{R}}\mathsf{y}^*\mathsf{x}^*$ and $\mathsf{b} \in \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \in \overline{N}$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{y}^* \leqslant \mathsf{x}^*$.

(ii) Suppose $\mathsf{a}^* \leqslant \mathsf{b}^*$; then by construction $b \leq a$. (a) Suppose $\mathsf{h}_{\mathsf{a}^*}(p) = 1$; then by (1), $h_a(\overline{p}) = 1$; so by $(\leq)$, $h_b(\overline{p}) = 1$; so by (1), $\mathsf{h}_{\mathsf{b}^*}(p) = 1$. Suppose $\mathsf{h}_{\mathsf{b}^*}(\overline{p}) = 1$; then by (1), $h_b(p) = 1$; so by $(\leq)$, $h_a(p) = 1$; so by (1), $\mathsf{h}_{\mathsf{a}^*}(\overline{p}) = 1$. (b.i) Suppose $\mathsf{b}^*\mathsf{Rxy}$ and $\mathsf{a}^* \notin \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \notin \overline{N}$; so by $(\leq)$, $a\overline{R}xy$ and by construction, $\mathsf{a}^*\mathsf{Rxy}$. Suppose $\mathsf{b}^*\mathsf{Rxy}$ and $\mathsf{a}^* \in \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \in \overline{N}$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (b.ii) Suppose $\mathsf{b}^*\mathsf{Ry}^*\mathsf{x}^*$ and $\mathsf{a}^* \notin \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \notin \overline{N}$; so by $(\leq)$, $a\overline{R}xy$ and by construction, $\mathsf{a}^*\mathsf{Ry}^*\mathsf{x}^*$. Suppose $\mathsf{b}^*\mathsf{Ry}^*\mathsf{x}^*$ and $\mathsf{a}^* \in \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \in \overline{N}$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{y}^* \leqslant \mathsf{x}^*$. (c.i) Suppose $\mathsf{a}^*\overline{\mathsf{R}}\mathsf{xy}$ and $\mathsf{b}^* \notin \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \notin N$; so by $(\leq)$, $bRxy$ and by construction, $\mathsf{b}^*\overline{\mathsf{R}}\mathsf{xy}$. Suppose $\mathsf{a}^*\overline{\mathsf{R}}\mathsf{xy}$ and $\mathsf{b}^* \in \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \in N$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (c.ii) Suppose $\mathsf{a}^*\overline{\mathsf{R}}\mathsf{y}^*\mathsf{x}^*$ and $\mathsf{b}^* \notin \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \notin N$; so by $(\leq)$, $bRxy$ and by construction, $\mathsf{b}^*\overline{\mathsf{R}}\mathsf{y}^*\mathsf{x}^*$. Suppose $\mathsf{a}^*\overline{\mathsf{R}}\mathsf{y}^*\mathsf{x}^*$ and $\mathsf{b}^* \in \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \in N$; so by $(\leq)$, $x \leq y$; so by construction $\mathsf{y}^* \leqslant \mathsf{x}^*$.

(iii) Suppose $\mathsf{a} \leqslant \mathsf{b}^*$; then by construction $a \leq^* b$. (a) Suppose $\mathsf{h}_{\mathsf{a}}(p) = 1$; then by (1), $h_a(p) = 1$; so by $(\leq^*)$, $h_b(\overline{p}) = 1$; so by (1), $\mathsf{h}_{\mathsf{b}^*}(p) = 1$. Suppose $\mathsf{h}_{\mathsf{b}^*}(\overline{p}) = 1$; then by (1), $h_b(p) = 1$; so by $(\leq^*)$, $h_a(\overline{p}) = 1$; so by (1), $\mathsf{h}_{\mathsf{a}}(\overline{p}) = 1$. (b.i) Suppose $\mathsf{b}^*\mathsf{Rxy}$ and $\mathsf{a} \notin \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \notin N$;

so by ($\leq^*$), $aRxy$ and by construction, aRxy. Suppose $\mathsf{b^*Rxy}$ and $\mathsf{a} \in \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \in N$; so by ($\leq^*$), $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (b.ii) Suppose $\mathsf{b^*Ry^*x^*}$ and $\mathsf{a} \notin \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \notin N$; so by ($\leq^*$), $aRxy$ and by construction, $\mathsf{aRy^*x^*}$. Suppose $\mathsf{b^*Ry^*x^*}$ and $\mathsf{a} \in \mathsf{N}$; then by construction, $b\overline{R}xy$ and $a \in N$; so by ($\leq$), $x \leq y$; so by construction $\mathsf{y^*} \leqslant \mathsf{x^*}$. (c.i) Suppose $\mathsf{a\overline{R}xy}$ and $\mathsf{b^*} \notin \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \notin N$; so by ($\leq^*$), $bRxy$ and by construction, $\mathsf{b^*\overline{R}xy}$. Suppose $\mathsf{a\overline{R}xy}$ and $\mathsf{b^*} \in \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \in N$; so by ($\leq^*$), $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (c.ii) Suppose $\mathsf{a\overline{R}y^*x^*}$ and $\mathsf{b^*} \notin \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \notin N$; so by ($\leq^*$), $bRxy$ and by construction, $\mathsf{b^*\overline{R}y^*x^*}$. Suppose $\mathsf{a\overline{R}y^*x^*}$ and $\mathsf{b} \in \overline{\mathsf{N}}$; then by construction, $a\overline{R}xy$ and $b \in N$; so by ($\leq^*$), $x \leq y$; so by construction $\mathsf{y^*} \leqslant \mathsf{x^*}$.

(iv) Suppose $\mathsf{a^*} \leqslant \mathsf{b}$; then by construction $a \leq^\sharp b$. (a) Suppose $\mathsf{h}_{\mathsf{a}^*}(p) = 1$; then by (1), $h_a(\overline{p}) = 1$; so by ($\leq^\sharp$), $h_b(p) = 1$; so by (1), $\mathsf{h}_{\mathsf{b}}(p) = 1$. Suppose $\mathsf{h}_{\mathsf{b}}(\overline{p}) = 1$; then by (1), $h_b(\overline{p}) = 1$; so by ($\leq^\sharp$), $h_a(p) = 1$; so by (1), $\mathsf{h}_{\mathsf{a}^*}(\overline{p}) = 1$. (b.i) Suppose bRxy and $\mathsf{a^*} \notin \mathsf{N}$; then by construction, $bRxy$ and $a \notin \overline{N}$; so by ($\leq^\sharp$), $a\overline{R}xy$ and by construction, $\mathsf{a^*Rxy}$. Suppose bRxy and $\mathsf{a^*} \in \mathsf{N}$; then by construction, $bRxy$ and $a \in \overline{N}$; so by ($\leq^\sharp$), $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (b.ii) Suppose $\mathsf{bRy^*x^*}$ and $\mathsf{a^*} \notin \mathsf{N}$; then by construction, $bRxy$ and $a \notin \overline{N}$; so by ($\leq^\sharp$), $a\overline{R}xy$ and by construction, $\mathsf{a^*Ry^*x^*}$. Suppose $\mathsf{bRy^*x^*}$ and $\mathsf{a^*} \in \mathsf{N}$; then by construction, $bRxy$ and $a \in \overline{N}$; so by ($\leq^\sharp$), $x \leq y$; so by construction $\mathsf{y^*} \leqslant \mathsf{x^*}$. (c.i) Suppose $\mathsf{a^*\overline{R}xy}$ and $\mathsf{b} \notin \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \notin \overline{N}$; so by ($\leq^\sharp$), $b\overline{R}xy$ and by construction, $\mathsf{b\overline{R}xy}$. Suppose $\mathsf{a^*\overline{R}xy}$ and $\mathsf{b} \in \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \in \overline{N}$; so by ($\leq^\sharp$), $x \leq y$; so by construction $\mathsf{x} \leqslant \mathsf{y}$. (c.ii) Suppose $\mathsf{a^*\overline{R}y^*x^*}$ and $\mathsf{b} \notin \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \notin \overline{N}$; so by ($\leq^\sharp$), $b\overline{R}xy$ and by construction, $\mathsf{b\overline{R}y^*x^*}$. Suppose $\mathsf{a^*\overline{R}y^*x^*}$ and $\mathsf{b} \in \overline{\mathsf{N}}$; then by construction, $aRxy$ and $b \in \overline{N}$; so by ($\leq^\sharp$), $x \leq y$; so by construction $\mathsf{y^*} \leqslant \mathsf{x^*}$.

($\leqslant^*$)

($\leqslant^\sharp$)

(NC) (i) Suppose $\mathsf{w} \in /\mathsf{N}/$; then $w \in /N/$. Say $\mathsf{w/R/yz}$; then by construction, $w/R/yz$; so by NC, $y = z$; so $\mathsf{y} = \mathsf{z}$; and similarly if $\mathsf{w}/\mathsf{R}/\mathsf{z}^*\mathsf{y}^*$. Say $\mathsf{y} = \mathsf{z}$; then $y = z$ so by NC, $w/R/yz$; so $\mathsf{w}/\mathsf{R}/\mathsf{yz}$; and similarly for y^* and z^*. (ii) Suppose $\mathsf{w}^* \in /\mathsf{N}/$; then $w \in \backslash N \backslash$. Say $\mathsf{w}^*/\mathsf{R}/\mathsf{yz}$; then by construction $w\backslash R\backslash yz$; so by NC $y = z$; so $\mathsf{y} = \mathsf{z}$; and similarly if $\mathsf{w}^*/\mathsf{R}/\mathsf{z}^*\mathsf{y}^*$. Say $\mathsf{y} = \mathsf{z}$; then $y = z$ so by NC, $w\backslash R\backslash yz$; so $\mathsf{w}^*/\mathsf{R}/\mathsf{yz}$; and similarly for y^* and z^*.

(CL) (i) Suppose $\mathsf{w} \in /\mathsf{N}/$; then by construction, $w \in /N/$; so by CL, $w \in \backslash N \backslash$; so by construction, $\mathsf{w} \in \backslash\mathsf{N}\backslash$. Suppose $\mathsf{w}^* \in /\mathsf{N}/$; then by construction, $w \in \backslash N \backslash$; so by CL, $w \in /N/$; so by construction, $\mathsf{w}^* \in \backslash\mathsf{N}\backslash$. (ii) Say $\mathsf{w} \in \mathsf{N}$; then by construction, $w \in N$. By construction, $\mathsf{h}_\mathsf{w}(/\mathsf{p}/) = 1$ iff $h_w(/p/) = 1$; by CL iff $h_w(\backslash p \backslash) = 1$; by construction iff $\mathsf{h}_\mathsf{w}(\backslash\mathsf{p}\backslash) = 1$. By construction, $\mathsf{h}_{\mathsf{w}^*}(/\mathsf{p}/) = 1$ iff $h_w(\backslash p \backslash) = 1$; by CL iff $h_w(/p/) = 1$; by construction iff $\mathsf{h}_{\mathsf{w}^*}(\backslash\mathsf{p}\backslash) = 1$.

(MC) Suppose $\mathsf{w} \in /\mathsf{N}/$ and wMx; then by construction, $w \in /N/$ and wMx; so by MC, $x \in /N/$; so by construction, $\mathsf{x} \in /\mathsf{N}/$. Suppose $\mathsf{w}^* \in /\mathsf{N}/$ and $\mathsf{w}^*\mathsf{M}\mathsf{x}^*$; then by construction, $w \in \backslash N \backslash$ and wMx; so by MC, $x \in \backslash N \backslash$; so by construction, $\mathsf{x}^* \in /\mathsf{N}/$. So $\langle \mathsf{W}, \mathsf{M}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \preccurlyeq, \mathsf{h} \rangle$ satisfies MC.

(CM) ($\leqslant$)

($\leqslant^*$) (i) Suppose $\mathsf{a} \leqslant^* \mathsf{b}$; then by construction $a \leq^* b$. Suppose bMc; then by construction, bMc; so by CM, there is some $y \leq^* c$ such that aMy; so by construction $\mathsf{y} \leqslant^* \mathsf{c}$ and aMy. Suppose aMc then by construction aMc so by CM there is some y, $c \leq^* y$ such that bMy; so by construction, $\mathsf{c} \leqslant^* \mathsf{y}$ and bMy.

(ii) Suppose $\mathsf{a}^* \leqslant^* \mathsf{b}^*$; then by construction $a \leq^\sharp b$. Suppose $\mathsf{b}^*\mathsf{M}\mathsf{c}^*$; then by construction, bMc; so by CM, there is some $y \leq^\sharp c$ such that aMy; so by construction $\mathsf{y}^* \leqslant^* \mathsf{c}^*$ and $\mathsf{a}^*\mathsf{M}\mathsf{y}^*$. Suppose $\mathsf{a}^*\mathsf{M}\mathsf{c}^*$ then by construction aMc so by CM there is some y, $c \leq^\sharp y$ such that bMy; so by construction, $\mathsf{c}^* \leqslant^* \mathsf{y}^*$ and $\mathsf{b}^*\mathsf{M}\mathsf{y}^*$.

(iii) Suppose $\mathsf{a}^* \leqslant^* \mathsf{b}$; then by construction $b \leq a$. Suppose bMc; then by construction, bMc; so by CM, there is some y, $c \leq y$ such that aMy; so by construction, $\mathsf{y}^* \leqslant^* \mathsf{c}$ and $\mathsf{a}^*\mathsf{M}\mathsf{y}^*$. Suppose $\mathsf{a}^*\mathsf{M}\mathsf{c}^*$ then by construction aMc so by CM there is some $y \leq c$ such that bMy; so by construction $\mathsf{c}^* \leqslant^* \mathsf{y}$ and bMy.

(iv) Suppose $\mathsf{a} \leqslant^* \mathsf{b}^*$; then by construction $a \leq b$. Suppose $\mathsf{b^*Mc^*}$; then by construction, bMc; so by CM, there is some $y \leq c$ such that aMy; so by construction $\mathsf{y} \leqslant^* \mathsf{c^*}$ and aMy. Suppose aMc then by construction aMc so by CM there is some y, $c \leq y$ such that bMy; so by construction, $\mathsf{c} \leqslant \mathsf{y}$ and bMy.

($\leqslant^\sharp$)

(ρ)

(σ) Suppose $\langle \mathsf{x}, \mathsf{y} \rangle \in \mathsf{M}$; then by construction, $\langle x, y \rangle \in M$; so by σ, $\langle y, x \rangle \in M$; so by construction $\langle \mathsf{y}, \mathsf{x} \rangle \in \mathsf{M}$. Suppose $\langle \mathsf{x^*}, \mathsf{y^*} \rangle \in \mathsf{M}$; then by construction, $\langle x, y \rangle \in M$; so by σ, $\langle y, x \rangle \in M$; so by construction $\langle \mathsf{y^*}, \mathsf{x^*} \rangle \in \mathsf{M}$. So $\langle \mathsf{W}, \mathsf{M}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \preceq, \mathsf{h} \rangle$ satisfies σ.

(τ)

(C^9_{10}) Suppose $\mathsf{a/R/bx}$ and xRcd; then by construction, $a/R/bx$ and $xRcd$; so by C^9_{10} there is a y such that $bRcy$ and $a/R/yd$ and a z such that $bRzd$ and $a/R/cz$; so by construction, there is a y such that bRcy and $\mathsf{a/R/yd}$, and a z such that bRzd and $\mathsf{a/R/cz}$.

Suppose $\mathsf{a/R/bx}$ and $\mathsf{xRc^*d^*}$; then by construction, $a/R/bx$ and $xRdc$; so by C^9_{10} there is a y such that $bRdy$ and $a/R/yc$ and a z such that $bRzc$ and $a/R/dz$; so by construction, there is a $\mathsf{y^*}$ such that $\mathsf{bRy^*d^*}$ and $\mathsf{a/R/c^*y^*}$, and a $\mathsf{z^*}$ such that $\mathsf{bRc^*z^*}$ and $\mathsf{a/R/z^*d^*}$.

Suppose $\mathsf{a/R/b^*x^*}$ and $\mathsf{x^*Rcd}$; then by construction, $a/R/xb$ and $x\overline{R}cd$; so by C^9_{10} there is a y such that $b\overline{R}cy$ and $a/R/yd$ and a z such that $b\overline{R}zd$ and $a/R/cz$; so by construction, there is a y such that $\mathsf{b^*Rcy}$ and $\mathsf{a/R/yd}$, and a z such that $\mathsf{b^*Rzd}$ and $\mathsf{a/R/cz}$.

Suppose $\mathsf{a/R/b^*x^*}$ and $\mathsf{x^*Rc^*d^*}$; then by construction, $a/R/xb$ and $x\overline{R}dc$; so by C^9_{10} there is a y such that $b\overline{R}dy$ and $a/R/yc$ and a z such that $b\overline{R}zc$ and $a/R/dz$; so by construction, there is a $\mathsf{y^*}$ such that $\mathsf{b^*Ry^*d^*}$ and $\mathsf{a/R/c^*y^*}$, and a $\mathsf{z^*}$ such that $\mathsf{b^*Rc^*z^*}$ and $\mathsf{a/R/z^*d^*}$. And similarly for the other cases.

(C11)

(C12) Suppose $\mathsf{a^*Rbc}$; then by construction, $a\overline{R}bc$; so by C12, there is some y, $a \leq^\sharp y$, $bRyc$ and some $z \leq a$, $c\overline{R}bz$; so by construction, $\mathsf{a^*} \leqslant \mathsf{y}$, bRyc, $\mathsf{a^*} \leqslant^* \mathsf{z}$ and $\mathsf{c\overline{R}bz}$.

Suppose $\mathsf{aRb^*c^*}$; then by construction, $aRcb$; so by C12, there is some y, $a \leq y$, $cRyb$ and some, z, $a \leq^* z$, $b\overline{R}cz$; so by construction, $\mathsf{a} \leqslant^* \mathsf{y^*}$, $\mathsf{c^*\overline{R}b^*y^*}$, $\mathsf{a} \leqslant \mathsf{z^*}$ and $\mathsf{b^*Rz^*c^*}$.

Suppose $\mathsf{a^*\overline{R}b^*c^*}$; then by construction, $aRcb$; so by C12, there is some y, $a \leq y$, $cRyb$ and some, z, $a \leq^* z$, $b\overline{R}cz$; so by construction, $\mathsf{y^*} \leqslant \mathsf{a^*}$, $\mathsf{c^*\overline{R}b^*y^*}$, $\mathsf{a^*} \leqslant^\sharp \mathsf{z^*}$ and $\mathsf{b^*Rz^*c^*}$.

And similarly in other cases.

L9.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{R4x} /P/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$ then $\Gamma' \vDash^*_{R4x} /P/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{R4x} /P/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$, but $\Gamma' \nvDash^*_{R4x} /P/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$. From the latter, by VR4x*, there is some $R4x$ interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma') = 1$ but $h_{m(s)}(/P/) = 0$ $\llbracket \langle m(s),\ m(t)\rangle \notin M,\ \langle m(r), m(s), m(t)\rangle \notin /R/,\ m(s) \neq m(t),\ \langle m(s), m(t)\rangle \notin \preceq \rrbracket$. But since $h_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $h_m(\Gamma) = 1$; so $h_m(\Gamma) = 1$ but $h_{m(s)}(/P/) = 0$ $\llbracket \langle m(s),\ m(t)\rangle \notin M,\ \langle m(r), m(s), m(t)\rangle \notin /R/,\ m(s) \neq m(t),\ \langle m(s), m(t)\rangle \notin \preceq \rrbracket$; so by VR4x*, $\Gamma \nvDash^*_{R4x} /P/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{R4x} /P/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$, then $\Gamma' \vDash^*_{R4x} /P/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NR4x} \mathcal{P}$ then $\Gamma \vDash^*_{R4x} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are without overlines and have subscript 0. Suppose $\Gamma \vdash^*_{NR4x} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \vDash^*_{R4x} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$. Then $\Gamma_1 = \{/A/_s\} \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$; so for any $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$, $h_m(\Gamma_1) = 1$ iff $h_{m(s)}(/A/) = 1$ $\llbracket \langle m(s),\ m(t)\rangle \in M,\ \langle m(r), m(s), m(t)\rangle \in /R/,\ m(s) = m(t),\ \langle m(s), m(t)\rangle \in \preceq \rrbracket$; so there is no $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma_1) = 1$ but $h_{m(s)}(/A/) = 0$ $\llbracket \langle m(s),\ m(t)\rangle \notin M,\ \langle m(r), m(s), m(t)\rangle \notin /R/,\ m(s) \neq m(t),\ \langle m(s), m(t)\rangle \notin \preceq \rrbracket$. So by VR4x*, $\Gamma_1 \vDash^*_{R4x} /A/_s \llbracket s.t,\ /r.s.t/,\ s \simeq t,\ s \preceq t \rrbracket$, where this is just to say, $\Gamma_1 \vDash^*_{R4x} \mathcal{P}_1$.

Assp: For any $i, 1 \le i < k, \Gamma_i \models^*_{R4x} \mathcal{P}_i$.

Show: $\Gamma_k \models^*_{R4x} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, ∧I, ∧E, ∨I, ∨E, ¬I, ¬E, ⊃I, ⊃E, →I, →E or, depending on the system, NI, NE, ≃I, ≃E, CL, then □I, □E, MC, AMρ, AMσ, AMτ, and AM$^9_{10}$, AM11, AM12, $\le$E, $\le^*$E, $\le^\sharp$E, $\le_R$E, $\le^*_R$E, $\le^\sharp_R$E, or $\preceq_M$E. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \models^*_{R4x} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

(∧I)

(∧E)

(∨I)

(∨E)

(¬I)

(¬E)

(⊃I)

(⊃E)

(→I) If $\mathcal{P}_k$ arises by →I, then the picture is like this,

$$\begin{array}{r|l|l} & & /s.t.u/ \\ & & \underline{A_t} \\ & & \\ i & & B_u \\ k & \multicolumn{2}{l}{/A \to B/_s} \end{array} \qquad \text{or} \qquad \begin{array}{r|l|l} & & /s.t.u/ \\ & & \underline{\overline{A}_u} \\ & & \\ i & & \overline{B}_t \\ k & \multicolumn{2}{l}{/A \to B/_s} \end{array}$$

where $i < k$, t and u do not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $/A \to B/_s$. In the first case, by assumption, $\Gamma_i \models^*_{R4x} B_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k \cup \{/s.t.u/, A_t\}$; so by L9.1, $\Gamma_k \cup \{/s.t.u/, A_t\} \models^*_{R4x} B_u$. Suppose $\Gamma_k \not\models^*_{R4x} /A \to B/_s$; then by VR4x*, there is an *R4x* interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/A \to B/) = 0$. From the latter, by HR4(→), there are $x, y \in W$ such that $m(s)/R/xy$ and either $h_x(A) = 1$ and $h_y(B) = 0$, or $h_y(\overline{A}) = 1$ and $h_x(\overline{B}) = 0$. But then by L9.0, there is an $R4x$ interpretation, $\langle \mathsf{W}, \mathsf{M}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \preccurlyeq, \mathsf{h}\rangle$ with $\mathsf{w}, \mathsf{w}^* \in \mathsf{W}$ corresponding to each $w \in W$ such that for any $/A/$, $\mathsf{h}_{\mathsf{w}}(/A/) = 1$ iff $h_w(/A/) = 1$ and $\mathsf{h}_{\mathsf{w}^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$. So, where $m(s) = w$ iff $\mathsf{m}(s) = \mathsf{w}$ (and the construction retains other relations on those worlds), it remains that $\mathsf{h}_{\mathsf{m}}(\Gamma_k) = 1$. In addition, by construction, if $m(s)/R/xy$ and $h_x(A) = 1$ but $h_y(B) = 0$, then $\mathsf{m}(s)/\mathsf{R}/\mathsf{xy}$ and $\mathsf{h}_{\mathsf{x}}(A) = 1$ but $\mathsf{h}_{\mathsf{y}}(B) = 0$. And if $m(s)/R/xy$ and $h_y(\overline{A}) = 1$ but $h_x(\overline{B}) = 0$, then $\mathsf{m}(s)/\mathsf{R}/\mathsf{y}^*\mathsf{x}^*$ and $\mathsf{h}_{\mathsf{y}^*}(A) = 1$ but $\mathsf{h}_{\mathsf{x}^*}(B) = 0$. Either way, then, there are $\mathsf{a}, \mathsf{b} \in \mathsf{W}$ such that $\mathsf{m}(s)/\mathsf{R}/\mathsf{ab}$ where $\mathsf{h}_{\mathsf{a}}(A) = 1$ and $\mathsf{h}_{\mathsf{b}}(B) = 0$. Now consider a map m' like m except that $\mathsf{m}'(t) = \mathsf{a}$ and $\mathsf{m}'(u) = \mathsf{b}$, and consider $\langle \mathsf{W}, \mathsf{M}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \preccurlyeq, \mathsf{h}\rangle_{\mathsf{m}'}$; since t and u do not appear in Γ_k it remains that $\mathsf{h}_{\mathsf{m}'}(\Gamma_k) = 1$; since $\mathsf{m}'(t) = \mathsf{a}$, $\mathsf{h}_{\mathsf{m}'(t)}(A) = 1$; and since $\mathsf{m}(s)/\mathsf{R}/\mathsf{ab}$, $\langle \mathsf{m}'(s), \mathsf{m}'(t), \mathsf{m}'(u)\rangle \in /\mathsf{R}/$; so $\mathsf{h}_{\mathsf{m}'}(\Gamma_k \cup \{A_t, /s.t.u/\}) = 1$; so by VR4X*, $\mathsf{h}_{\mathsf{m}'(u)}(B) = 1$; so $\mathsf{h}_{\mathsf{b}}(B) = 1$. Reject the assumption: $\Gamma_k \models^*_{\mathit{R4x}} /A \to B/_s$, which is to say, $\Gamma_k \models^*_{\mathit{R4x}} \mathcal{P}_k$. And similarly in the other case.

(→E)

(NI)

(NE)

(≃I)

(≃E)

(CL) If $\mathcal{P}_k$ arises by CL, then the picture is like this,

$$\begin{array}{r|l} i & /n/[s] \\ j & /\!\!/A/\!\!/_s \\ & \\ k & \backslash\!\!\backslash A\backslash\!\!\backslash_s \end{array} \qquad \text{or} \qquad \begin{array}{r|l} i & /n/[s] \\ & \\ k & \backslash n\backslash[s] \end{array}$$

where, for the first case, $i, j < k$ and $\mathcal{P}_k$ is $\backslash\!\!\backslash A\backslash\!\!\backslash_s$. Where this rule is included in $NR4x$, R4x includes constraint CL along with NC. By assumption, $\Gamma_i \models^*_{\mathit{R4x}} /n/[s]$ and $\Gamma_j \models^*_{\mathit{R4x}} /\!\!/A/\!\!/_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{\mathit{R4x}} /n/[s]$ and $\Gamma_k \models^*_{\mathit{R4x}} /\!\!/A/\!\!/_s$. Suppose $\Gamma_k \not\models^*_{\mathit{R4x}} \backslash\!\!\backslash A\backslash\!\!\backslash_s$; then by VR4X*, there is an $R4x$ interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\backslash\!\!\backslash A\backslash\!\!\backslash) = 0$; since $h_m(\Gamma_k) = 1$, by VR4X*, $m(s) \in /N/$ and $h_{m(s)}(/\!\!/A/\!\!/) = 1$.

Now, by induction on the number of operators in A we show that for any $x \in /N/$, $h_x(/\!\!/A/\!\!/) = h_x(\backslash\!\!\backslash A\backslash\!\!\backslash)$.

Basis: $/\!\!/A/\!\!/$ is a parameter $/\!\!/p/\!\!/$. Suppose $x \in /N/$; then by CL, $h_x(/\!\!/p/\!\!/) = h_x(\backslash\!\!\backslash p\backslash\!\!\backslash)$.

Assp: For any i, $0 \leq i < k$, if A has i operator symbols then for any $x \in /N/$, $h_x(/\!\!/A/\!\!/) = h_x(\backslash\!\!\backslash A\backslash\!\!\backslash)$.

Show: If A has k operator symbols then for any $x \in /N/$, $h_x(/\!\!/A/\!\!/) = h_x(\backslash\!\!\backslash A\backslash\!\!\backslash)$.

If A has k operator symbols, then it is of the form $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$, or $\Box P$ where P and Q have $< k$ operator symbols. Suppose $x \in /N/$.

($\neg$) A is $\neg P$. By HR4($\neg$), $h_x(/\!\!/\neg P/\!\!/) = 1$ iff $h_x(\backslash\!\!\backslash P\backslash\!\!\backslash) = 0$; by assumption, iff $h_x(/\!\!/P/\!\!/) = 0$; by HR4($\neg$) iff $h_x(\backslash\!\!\backslash \neg P\backslash\!\!\backslash) = 1$.

($\wedge$) A is $P \wedge Q$. By HR4($\wedge$), $h_x(/\!\!/P \wedge Q/\!\!/) = 1$ iff $h_x(/\!\!/P/\!\!/) = 1$ and $h_x(/\!\!/Q/\!\!/) = 1$; by assumption iff $h_x(\backslash\!\!\backslash P\backslash\!\!\backslash) = 1$ and $h_x(\backslash\!\!\backslash Q\backslash\!\!\backslash) = 1$; by HR4($\wedge$), iff $h_x(\backslash\!\!\backslash P \wedge Q\backslash\!\!\backslash) = 1$.

($\vee$)

($\rightarrow$) A is $P \rightarrow Q$. Suppose $h_x(/\!\!/P \rightarrow Q/\!\!/) = 1$ but $h_x(\backslash\!\!\backslash P \rightarrow Q\backslash\!\!\backslash) = 0$. From the latter, by HR4($\rightarrow$), there are $y, z \in W$ such that $x\backslash\!\!\backslash R\backslash\!\!\backslash yz$ and $h_y(P) = 1$ but $h_z(Q) = 0$, or $h_z(\overline{P}) = 1$ but $h_y(\overline{Q}) = 0$. Since $x \in /N/$, by CL $x \in \backslash N\backslash$; so that $x \in /\!\!/N/\!\!/$ and $x \in \backslash\!\!\backslash N\backslash\!\!\backslash$; so with $x\backslash\!\!\backslash R\backslash\!\!\backslash yz$, by NC $y = z$, and with NC again $x/\!\!/R/\!\!/yz$. So from $h_x(/\!\!/P \rightarrow Q/\!\!/) = 1$, it is not the case that $h_y(P) = 1$ but $h_z(Q) = 0$, or $h_z(\overline{P}) = 1$ but $h_y(\overline{Q}) = 0$. Reject the assumption: it is not the case that $h_x(/\!\!/P \rightarrow Q/\!\!/) = 1$ but $h_x(\backslash\!\!\backslash P \rightarrow Q\backslash\!\!\backslash) = 0$.

($\Box$) A is $\Box P$. Suppose $h_x(/\!\!/\Box P/\!\!/) = 1$ but $h_x(\backslash\!\!\backslash \Box P\backslash\!\!\backslash) = 0$. From the latter, by HR4($\Box$), there is some $y \in W$ such that xMy and $h_y(\backslash\!\!\backslash P\backslash\!\!\backslash) = 0$; but since $x \in /N/$, by MC, $y \in /N/$; so by assumption, $h_y(/\!\!/P/\!\!/) = 0$; so by HR4($\Box$), $h_x(/\!\!/\Box P/\!\!/) = 0$. This is impossible; reject the assumption: it is not the case that $h_x(/\!\!/\Box P/\!\!/) = 1$ but $h_x(\backslash\!\!\backslash \Box P\backslash\!\!\backslash) = 0$.

For any A and $x \in /N/$, $h_x(/\!\!/A/\!\!/) = h_x(\backslash\!\!\backslash A\backslash\!\!\backslash)$

So, returning to the main case, $h_{m(s)}(\backslash\!\!\backslash A\backslash\!\!\backslash) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{R4x} \backslash\!\!\backslash A\backslash\!\!\backslash_s$; which is to say, $\Gamma_k \models^*_{R4x} \mathcal{P}_k$. The other case is straightforward.

(□I)

(□E)

(MC)

(AMρ)

(AMσ)

(AMτ)

(AM$^{9}_{10}$)

(AM11) If $\mathcal{P}_k$ arises by AM11, then the picture is like this,

$$\begin{array}{r|l|l} i & /s.t.u/ & \\ & & /s.t.y/ \\ & & \underline{y.t.u} \\ & & \\ & & \\ j & & /\!\!/A/\!\!/_w \\ k & /\!\!/A/\!\!/_w & \end{array} \qquad \text{or} \qquad \begin{array}{r|l|l} i & /s.t.u/ & \\ & & /s.y.u/ \\ & & \overline{\underline{y.t.u}} \\ & & \\ & & \\ j & & /\!\!/A/\!\!/_w \\ k & /\!\!/A/\!\!/_w & \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $/\!\!/A/\!\!/_w$. Where this rule is included in *NR4x*, R4x includes constraint C11. By assumption, in both cases, $\Gamma_i \models^*_{R4x} /s.t.u/$ and $\Gamma_j \models^*_{R4x} /\!\!/A/\!\!/_w$; but, in the left-hand case, by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{/s.t.y/, y.t.u\}$; so by L9.1, $\Gamma_k \models^*_{R4x} /s.t.u/$ and $\Gamma_k \cup \{/s.t.y/, y.t.u\} \models^*_{R4x} /\!\!/A/\!\!/_w$. Suppose $\Gamma_k \not\models^*_{R4x} /\!\!/A/\!\!/_w$; then by VR4x*, there is an *R4x* interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(w)}(/\!\!/A/\!\!/) = 0$; since $h_m(\Gamma_k) = 1$, by VR4x*, $\langle m(s), m(t), m(u)\rangle \in /R/$; and by C11, if $a/R/bc$ then there is a y such that $a/R/by$ and $yRbc$ and a z such that $a/R/zc$ and $z\overline{R}bc$; so there is a $v \in W$ such that $m(s)/R/m(t)v$ and $vRm(t)m(u)$; consider a map m' like m except that $m'(y) = v$, and consider $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_{m'}$; since y does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$, $m'(t) = m(t)$, $m'(y) = v$ and $m'(u) = m(u)$, $\langle m'(s), m'(t), m'(y)\rangle \in /R/$ and $\langle m'(y), m'(t), m'(u)\rangle \in R$; so $h_{m'}(\Gamma_k \cup \{/s.t.y/, y.t.u\}) = 1$; so by VR4x*, $h_{m'(w)}(/\!\!/A/\!\!/) = 1$. But since $y \neq w$, $m'(w) = m(w)$; so $h_{m(w)}(/\!\!/A/\!\!/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{R4x} /\!\!/A/\!\!/_w$, which is to say, $\Gamma_k \models^*_{R4x} \mathcal{P}_k$. And similarly for the right-hand case.

(AM12)

($\leq$E) If $\mathcal{P}_k$ arises by $\leq$E, then the picture is like this,

$$\begin{array}{r|l} i & a \leq b \\ j & A_a \\ & \\ k & A_b \end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is A_b. By assumption, $\Gamma_i \vDash^*_{R4x} a \leq b$ and $\Gamma_j \vDash^*_{R4x} A_a$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \vDash^*_{R4x} a \leq b$ and $\Gamma_k \vDash^*_{R4x} A_a$. Suppose $\Gamma_k \nvDash^*_{R4x} A_b$; then by VR4x*, there is an $R4x$ interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(b)}(A) = 0$; since $h_m(\Gamma_k) = 1$, by VR4x*, $\langle m(a), m(b)\rangle \in \leq$ and $h_{m(a)}(A) = 1$.

Now, by induction on the number of operators in A, we show that for any x, $y \in W$, if $x \leq y$, then (i) if $h_x(A) = 1$ then $h_y(A) = 1$, and (ii) if $h_y(\overline{A}) = 1$ then $h_x(\overline{A}) = 1$.

Basis: A is a parameter p. Suppose $x \leq y$. (i) Suppose $h_x(A) = 1$; then $h_x(p) = 1$; but since $x \leq y$, by ($\preceq$), $h_y(p) = 1$; so $h_y(A) = 1$. (ii) Suppose $h_y(\overline{A}) = 1$; then $h_y(\overline{p}) = 1$; but since $x \leq y$, by ($\preceq$), $h_x(\overline{p}) = 1$; so $h_x(\overline{A}) = 1$.

Assp: For any i, $0 \leq i < k$, if A has i operators, then for any x, $y \in W$, if $x \leq y$, then if $h_x(A) = 1$ then $h_y(A) = 1$, and if $h_y(\overline{A}) = 1$ then $h_x(\overline{A}) = 1$.

Show: If A has k operators, then for any x, $y \in W$, if $x \leq y$, then (i) if $h_x(A) = 1$ then $h_y(A) = 1$, and (ii) if $h_y(\overline{A}) = 1$ then $h_x(\overline{A}) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$, or $\Box P$, where P and Q have $< k$ operators. Suppose $x \leq y$.

($\neg$) A is $\neg P$. (i) Suppose $h_x(A) = 1$; then $h_x(\neg P) = 1$; so by HR4x($\neg$), $h_x(\overline{P}) = 0$; so by assumption, $h_y(\overline{P}) = 0$; so by HR4x($\neg$), $h_y(\neg P) = 1$, which is to say, $h_y(A) = 1$. (ii) Suppose $h_y(\overline{A}) = 1$; then $h_y(\overline{\neg P}) = 1$; so by HR4x($\neg$), $h_y(P) = 0$; so by assumption, $h_x(P) = 0$; so by HR4x($\neg$), $h_x(\overline{\neg P}) = 1$, which is to say, $h_x(\overline{A}) = 1$.

($\wedge$) A is $P \wedge Q$. (i) Suppose $h_x(A) = 1$; then $h_x(P \wedge Q) = 1$; so by HR4x($\wedge$), $h_x(P) = 1$ and $h_x(Q) = 1$; so by assumption, $h_y(P) = 1$ and $h_y(Q) = 1$; so by HR4x($\wedge$), $h_y(P \wedge Q) = 1$, which is to say $h_y(A) = 1$. (ii) Suppose $h_y(\overline{A}) = 1$; then

$h_y(\overline{P \wedge Q}) = 1$; so by HR4x($\wedge$), $h_y(\overline{P}) = 1$ and $h_y(\overline{Q}) = 1$; so by assumption, $h_x(\overline{P}) = 1$ and $h_x(\overline{Q}) = 1$; so by HR4x($\wedge$), $h_x(\overline{P \wedge Q}) = 1$, which is to say $h_x(\overline{A}) = 1$.

($\vee$)

($\rightarrow$) A is $P \rightarrow Q$. (i) Suppose $h_x(A) = 1$ but $h_y(A) = 0$; then $h_x(P \rightarrow Q) = 1$ and $h_y(P \rightarrow Q) = 0$; then by HR4x($\rightarrow$), there are some w, $z \in W$ such that $yRwz$ and (1) $h_w(P) = 1$ but $h_z(Q) = 0$, or (2) $h_z(\overline{P}) = 1$ but $h_w(\overline{Q}) = 0$. We consider these in two cases: (a) $x \notin N$; then since $yRwz$ and $x \leq y$, by ($\preceq$), $xRwz$. Suppose (1): $h_w(P) = 1$ but $h_z(Q) = 0$; then since $h_w(P) = 1$, $h_x(P \rightarrow Q) = 1$, and $xRwz$, by HR4x($\rightarrow$), $h_z(Q) = 1$. This is impossible. Suppose (2): $h_z(\overline{P}) = 1$ but $h_w(\overline{Q}) = 0$; then since $h_z(\overline{P}) = 1$, $h_x(P \rightarrow Q) = 1$, and $xRwz$, by HR4x($\rightarrow$), $h_w(\overline{Q}) = 1$. This is impossible. (b) $x \in N$; then since $yRwz$ and $x \leq y$, by ($\preceq$), $w \leq z$. Suppose (1): $h_w(P) = 1$ but $h_z(Q) = 0$; then since $x \in N$ and $w = w$, by NC, $xRww$; so since $h_w(P) = 1$, $h_x(P \rightarrow Q) = 1$, and $xRww$, by HR4x($\rightarrow$), $h_w(Q) = 1$; but since $w \leq z$, by assumption, $h_z(Q) = 1$. This is impossible; reject the assumption: $h_y(P \rightarrow Q) = 1$, which is to say $h_y(A) = 1$. Suppose (2): $h_z(\overline{P}) = 1$ but $h_w(\overline{Q}) = 0$; then since $x \in N$ and $z = z$, by NC, $xRzz$; so since $h_z(\overline{P}) = 1$, $h_x(P \rightarrow Q) = 1$, and $xRzz$, by HR4x($\rightarrow$), $h_z(\overline{Q}) = 1$; but since $w \leq z$, by assumption, $h_w(\overline{Q}) = 1$. This is impossible; reject the assumption: if $h_x(A) = 1$, then $h_y(A) = 1$.

And similarly for (ii).

($\Box$) A is $\Box P$. (i) Suppose $h_x(A) = 1$; then $h_x(\Box P) = 1$. Suppose $h_y(\Box P) = 0$; then by HR4x($\Box$), there is some $w \in W$ such that yMw and $h_w(P) = 0$; since $x \leq y$ and yMw, by the constraint on modal access (CM), there is some $v \leq w$ such that xMv; since $v \leq w$ and $h_w(P) = 0$, by assumption, $h_v(P) = 0$; so by HR4x($\Box$), $h_x(\Box P) = 0$. This is impossible; reject the assumption: $h_y(\Box P) = 1$, which is to say $h_y(A) = 1$.

(ii) Suppose $h_y(\overline{A}) = 1$; then $h_y(\overline{\Box P}) = 1$; suppose $h_x(\overline{\Box P}) = 0$; then by $HR4x(\Box)$, there is some $w \in W$ such that xMw and $h_w(\overline{P}) = 0$; since $x \leq y$ and xMw, by CM, there is some $v \geq w$ such that yMv; since $w \leq v$ and $h_w(\overline{P}) = 0$, by assumption, $h_v(\overline{P}) = 0$; so by HR4x($\Box$), $h_y(\overline{\Box P}) = 0$. This is impossible; reject the assumption: $h_x(\overline{\Box P}) = 1$, which is to say $h_x(\overline{A}) = 1$.

For any A and any $x, y \in W$, if $x \leq y$, then (i) if $h_x(A) = 1$ then $h_y(A) = 1$, and (ii) if $h_y(\overline{A}) = 1$ then $h_x(\overline{A}) = 1$.

So, returning to the case for ($\leq$E), $h_{m(b)}(A) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{R4x} A_b$, which is to say, $\Gamma_k \models^*_{R4x} \mathcal{P}_k$.

($\leq^*$E)

($\leq^\sharp$E)

($\leq_R$E) If $\mathcal{P}_k$ arises by $\leq_R$E, then the picture is like this,

$$\begin{array}{r|l} h & a \leq b \\ i & b.x.y \\ j & n[a] \\ & \\ k & x \leq y \end{array} \qquad \begin{array}{r|l} h & a \leq b \\ i & \overline{a.x.y} \\ j & \overline{n}[b] \\ & \\ k & x \leq y \end{array} \qquad \begin{array}{r|l} h & a \leq b \\ i & b.x.y \\ j & {\sim}n[a] \\ & \\ k & a.x.y \end{array} \qquad \begin{array}{r|l} h & a \leq b \\ i & \overline{a.x.y} \\ j & {\sim}\overline{n}[b] \\ & \\ k & \overline{b.x.y} \end{array}$$

where $h, i, j < k$. In the third case $\mathcal{P}_k$ is $a.x.y$. By assumption, $\Gamma_h \models^*_{R4x} a \leq b$, $\Gamma_i \models^*_{R4x} b.x.y$ and $\Gamma_j \models^*_{R4x} {\sim}n[a]$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L9.1, $\Gamma_k \models^*_{R4x} a \leq b$, $\Gamma_k \models^*_{R4x} b.x.y$ and $\Gamma_k \models^*_{R4x} {\sim}n[a]$. Suppose $\Gamma_k \not\models^*_{R4x} a.x.y$; then by VR4x*, there is an $R4x$ interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $\langle m(a), m(x), m(y)\rangle \notin R$; since $h_m(\Gamma_k) = 1$, by VR4x*, $\langle m(a), m(b)\rangle \in \leq$, $\langle m(b), m(x), m(y)\rangle \in R$ and $h_{m(\phi)}[{\sim}n(a)] = 1$, so that $h_{m(\phi)}[n(a)] = 0$ and $m(a) \notin N$; so with ($\preceq$), $\langle m(a), m(x), m(y)\rangle \in R$. This is impossible; reject the assumption. And similarly in other the cases.

($\leq^*_R$E)

($\leq^\sharp_R$E)

($\preceq_M$E) If $\mathcal{P}_k$ arises by $\preceq_M$E, then the picture is like this,

$$\begin{array}{r|l} h & a \preceq b \\ i & b.c \\ & \vert\; t \preceq c \\ & \vert\; \underline{a.t} \\ & \vert \\ & \vert \\ j & \vert\; /A/_w \\ k & /A/_w \end{array} \qquad \text{or} \qquad \begin{array}{r|l} h & a \preceq b \\ i & a.c \\ & \vert\; c \preceq t \\ & \vert\; \underline{b.t} \\ & \vert \\ & \vert \\ j & \vert\; /A/_w \\ k & /A/_w \end{array}$$

where $h, i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not w, and $\mathcal{P}_k$ is $/A/_w$.

Where this rule is included in *NR4x*, R4x includes CM. In the first case, by assumption, $\Gamma_h \models^*_{R4x} a \preceq b$, $\Gamma_i \models^*_{R4x} b.c$ and $\Gamma_j \models^*_{R4x} /A/_w$; but by the nature of access, $\Gamma_h \subseteq \Gamma_k$, $\Gamma_i \subseteq \Gamma_k$, and $\Gamma_j \subseteq \Gamma_k \cup \{t \preceq c, a.t\}$; so by L9.1, $\Gamma_k \models^*_{R4x} a \preceq b$, $\Gamma_k \models^*_{R4x} b.c$ and $\Gamma_k \cup \{t \preceq c, a.t\} \models^*_{R4x} /A/_w$. Suppose $\Gamma_k \not\models^*_{R4x} /A/_w$; then by VR4x*, there is an *R4x* interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(w)}(/A/) = 0$. Since $h_m(\Gamma_k) = 1$, by VR4x*, $\langle m(a), m(b)\rangle \in \preceq$ and $\langle m(b), m(c)\rangle \in M$ so by CM, there is some y such that $\langle y, m(c)\rangle \in \preceq$ and $\langle m(a), y\rangle \in M$. Consider a map m' like m except that $m'(t) = y$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; since $m'(t) = y$ and other values are unchanged, $\langle m'(t), m'(c)\rangle \in \preceq$ and $\langle m'(a), m'(t)\rangle \in M$; so $h_{m'}(\Gamma_k \cup \{t \preceq c, a.t\}) = 1$; so by VR4x*, $h_{m'(w)}(/A/) = 1$; and since $m'(w) = m(w)$, $h_{m(w)}(/A/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{R4x} /A/_w$, where this is to say, $\Gamma_k \models^*_{R4x} \mathcal{P}_k$.

For any i, $\Gamma_i \models^*_{R4x} \mathcal{P}_i$.

THEOREM 9.2 *NR4x is complete: if* $\Gamma \models_{R4x} A$ *then* $\Gamma \vdash_{NR4x} A$.

Suppose $\Gamma \models_{R4x} A$; then $\Gamma_0 \models^*_{R4x} A_0$; we show that $\Gamma_0 \vdash^*_{NR4x} A_0$. As usual, this reduces to the standard notion. For the following, fix on some particular *R*4*x*. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NR4x} /A/_s$ and $\Gamma \vdash^*_{NR4x} \backslash\neg A\backslash_s$.

L9.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NR4x} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

Reasoning as in L7.2.

L9.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1$ $\mathcal{P}_2 \ldots$ In addition, there is an enumeration of these formulas with expressions of the sort $s.t$ and $s.t.u$ and with pairs of the sort $s.t.u$,$u.v.w$ and $s \preceq t, u.v$.

Proof by construction.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NR4x} /A/_s$ or $\Gamma \vdash^*_{NR4x} \backslash\neg A\backslash_s$.

SGT Γ is a SCAPEGOAT set for $\rightarrow$ iff for every formula of the form $/\neg(A \rightarrow B)/_s$, if $\Gamma \vdash^*_{NR4x} /\neg(A \rightarrow B)/_s$ then there are some t, u such that $\Gamma \vdash^*_{NR4x} \backslash s.t.u\backslash$, $\Gamma \vdash^*_{NR4x} A_t$ and $\Gamma \vdash^*_{NR4x} \overline{\neg B}_u$.

Γ is a SCAPEGOAT set for $\Box$ iff for every formula of the form $/\neg\Box A/_s$, if $\Gamma \vdash^*_{NR4x} /\neg\Box A/_s$ then there is some t such that $\Gamma \vdash^*_{NR4x} s.t$ and $\Gamma \vdash^*_{NR4x} /\neg A/_t$.

Γ is a SCAPEGOAT set for C9/C10 iff (i) for every access pair $/s.t.u/$, $u.v.w$, if $\Gamma \vdash^*_{NR4x} /s.t.u/$ and $\Gamma \vdash^*_{NR4x} u.v.w$, then there is some y such that $\Gamma \vdash^*_{NR4x} t.v.y$ and $\Gamma \vdash^*_{NR4x} /s.y.w/$, and there is some z such that $\Gamma \vdash^*_{NR4x} t.z.w$ and $\Gamma \vdash^*_{NR4x} /s.v.z/$; and (ii) for every access pair $/s.u.t/, \overline{u.v.w}$, if $\Gamma \vdash^*_{NR4x} /s.u.t/$ and $\Gamma \vdash^*_{NR4x} \overline{u.v.w}$, then there is some y such that $\Gamma \vdash^*_{NR4x} \overline{t.v.y}$ and $\Gamma \vdash^*_{NR4x} /s.y.w/$, and there is some z such that $\Gamma \vdash^*_{NR4x} \overline{t.z.w}$ and $\Gamma \vdash^*_{NR4x} /s.v.z/$.

Γ is a SCAPEGOAT set for C11 iff for every access relation $/s.t.u/$, if $\Gamma \vdash^*_{NR4x} /s.t.u/$ then there is some y such that $\Gamma \vdash^*_{NR4x} /s.t.y/$ and $\Gamma \vdash^*_{NR4x} y.t.u$, and there is some z such that $\Gamma \vdash^*_{NR4x} /s.z.u/$ and $\Gamma \vdash^*_{NR4x} \overline{z.t.u}$.

Γ is a SCAPEGOAT set for C12 iff for every access relation $s.t.u$, if $\Gamma \vdash^*_{NR4x} s.t.u$ then there is some y such that $\Gamma \vdash^*_{NR4x} y \geq s$ and $\Gamma \vdash^*_{NR4x} t.y.u$, and there is some z such that $\Gamma \vdash^*_{NR4x} z \geq^* s$ and $\Gamma \vdash^*_{NR4x} \overline{u.t.z}$; and if $\Gamma \vdash^*_{NR4x} \overline{s.t.u}$ then there is some y such that $\Gamma \vdash^*_{NR4x} y \geq^\sharp s$ and $\Gamma \vdash^*_{NR4x} t.y.u$, and there is some z such that $\Gamma \vdash^*_{NR4x} z \leq s$ and $\Gamma \vdash^*_{NR4x} \overline{u.t.z}$

Γ is a SCAPEGOAT set for CM iff for every pair $s \preceq t$, $t.u$, if $\Gamma \vdash^*_{NR4x} s \preceq t$ and $\Gamma \vdash^*_{NR4x} t.u$ there is some y such that $\Gamma \vdash^*_{NR4x} y \preceq u$ and $\Gamma \vdash^*_{NR4x} s.y$; and for every $s \preceq t$, $s.u$, if $\Gamma \vdash^*_{NR4x} s \preceq t$ and $\Gamma \vdash^*_{NR4x} s.u$ there is some y such that $\Gamma \vdash^*_{NR4x} u \preceq y$ and $\Gamma \vdash^*_{NR4x} t.y$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L9.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the formulas, together with all the access relations $s.t$ and $s.t.u$, and access pairs $s.t.u, u.v.w$ if C9/C10 is in $R4x$ and $s \preceq t, u.v$ if CM is in $R4x$; let $\mathcal{E}_0$ be this enumeration. Then for the first expression $\mathcal{P}$ in $\mathcal{E}_{i-1}$ such that all its subscripts are 0 or introduced in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $\mathcal{P}$, and set,

$$\begin{array}{ll}
\Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \cup \{\mathcal{P}\} \text{ is inconsistent} \\
\Omega_{i^*} = \Omega_{i-1} \cup \{\mathcal{P}\} & \text{if } \Omega_{i-1} \cup \{\mathcal{P}\} \text{ is consistent}
\end{array}$$

and

$$\begin{array}{rll}
\rightarrow: & \Omega_i = \Omega_{i^*} \cup \{\backslash s.y.z\backslash, P_y, \overline{\neg Q}_z\} & \text{if } \mathcal{P} \text{ is of the form } /\neg(P \rightarrow Q)/_s \\
\Box: & \Omega_i = \Omega_{i^*} \cup \{s.y, /\neg P/_y\} & \text{if } \mathcal{P} \text{ is of the form } /\neg\Box P/_s \\
\text{C}^9_{10}: & \Omega_i = \Omega_{i^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\} & \text{if } \mathcal{P} \text{ is of the form } /s.t.u/, u.v.w \\
 & \Omega_i = \Omega_{i^*} \cup \{\overline{t.v.y}, /s.y.w/, \overline{t.z.w}, /s.v.z/\} & \text{if } \mathcal{P} \text{ is of the form } /s.u.t/, \overline{u.v.w} \\
\text{C11}: & \Omega_i = \Omega_{i^*} \cup \{/s.t.y/, y.t.u, /s.z.u/, \overline{z.t.u}\} & \text{if } \mathcal{P} \text{ is of the form } /s.t.u/ \\
\text{C12}: & \Omega_i = \Omega_{i^*} \cup \{y \geq s, t.y.u, z \geq^* s, \overline{u.t.z}\} & \text{if } \mathcal{P} \text{ is of the form } s.t.u \\
 & \Omega_i = \Omega_{i^*} \cup \{y \geq^\sharp s, t.y.u, z \leq s, \overline{u.t.z}\} & \text{if } \mathcal{P} \text{ is of the form } \overline{s.t.u} \\
\text{CM}: & \Omega_i = \Omega_{i^*} \cup \{y \preceq u, s.y\} & \text{if } \mathcal{P} \text{ is of the form } s \preceq t,\ t.u \\
 & \Omega_i = \Omega_{i^*} \cup \{u \preceq y, t.y\} & \text{if } \mathcal{P} \text{ is of the form } s \preceq t,\ s.u
\end{array}$$

-where y, z are the first subscripts not included in Ω_{i^*}

and

$$\begin{array}{ll}
\Omega_i = \Omega_{i^*} & \text{otherwise}
\end{array}$$

then

$$\Gamma' = \bigcup_{i \geq 0} \Omega_i$$

Note that there are always sure to be subscripts y, z not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s appears in Γ'; then there is some Ω_i in which it is first appears; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L9.4 For any s included in Γ', Γ' is s-maximal.

Reasoning as in L7.4.

L9.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$, (iii) $\Omega_{k^*} \cup \{\backslash s.y.z\backslash, P_y, \overline{\neg Q}_z\}$, (iv) $\Omega_{k^*} \cup \{s.y, /\neg P/_y\}$, (v.a) $\Omega_{k^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\}$, (v.b) $\Omega_{k^*} \cup \{\overline{t.v.y}, /s.y.w/, \overline{t.z.w}, /s.v.z/\}$, (vi) $\Omega_{k^*} \cup \{/s.t.y/, y.t.u, /s.z.u/, \overline{z.t.u}\}$, (vii.a) $\Omega_{k^*} \cup \{y \geq s, t.y.u, z \geq^* s, \overline{u.t.z}\}$, (vii.b) $\Omega_{k^*} \cup \{y \geq^\sharp s, t.y.u, z \leq s, \overline{u.t.z}\}$, (viii.a) $\Omega_{k^*} \cup \{y \preceq u, s.y\}$ or (viii.b) $\Omega_{k^*} \cup \{u \preceq y, t.y\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{\mathcal{P}\}$. Then by construction, $\Omega_{k-1} \cup \{\mathcal{P}\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{\backslash s.y.z\backslash, P_y, \overline{\neg Q}_z\}$. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg(P \rightarrow Q)/_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there is some A_v such that $\Omega_{k^*} \cup \{\backslash s.y.z\backslash, P_y, \overline{\neg Q}_z\} \vdash^*_{NR4x} /A/_v$ and $\Omega_{k^*} \cup \{\backslash s.y.z\backslash, P_y, \overline{\neg Q}_z\} \vdash^*_{NR4x} \backslash\neg A\backslash_v$. So reason as follows,

$$\begin{array}{r|ll}
1 & \Omega_{k^*} & \\
2 & \quad\vert\ \backslash s.y.z\backslash & \text{A } (g, \rightarrow\text{I}) \\
3 & \quad\vert\ \underline{P_y} & \text{A } (g, \rightarrow\text{I}) \\
4 & \quad\vert\quad\vert\ \underline{\overline{\neg Q}_z} & \text{A } (c, \neg\text{E}) \\
5 & \quad\vert\quad\vert\ /A/_v & \text{from } \Omega_{k^*} \cup \{\backslash s.y.z\backslash, P_y, \overline{\neg Q}_z\} \\
6 & \quad\vert\quad\vert\ \backslash\neg A\backslash_v & \text{from } \Omega_{k^*} \cup \{\backslash s.y.z\backslash, P_y, \overline{\neg Q}_z\} \\
7 & \quad\vert\ Q_z & \text{4-6 } \neg\text{E} \\
8 & \backslash P \rightarrow Q\backslash_s & \text{2-7 } \rightarrow\text{I}
\end{array}$$

where, by construction, y and z are not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NR4x} \backslash P \rightarrow Q\backslash_s$; but $/\neg(P \rightarrow Q)/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NR4x} /\neg(P \rightarrow Q)/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(iv) Suppose Ω_k is $\Omega_{k^*} \cup \{s.y, /\neg P/_y\}$. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg\Box P/_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there is some A_u such that $\Omega_{k^*} \cup \{s.y, /\neg P/_y\} \vdash^*_{NR4x} /A/_u$ and $\Omega_{k^*} \cup \{s.y, /\neg P/_y\} \vdash^*_{NR4x} \backslash\neg A\backslash_u$. So reason as follows,

$$\begin{array}{r|ll}
1 & \Omega_{k^*} & \\
2 & \quad\vert\ \underline{s.y} & \text{A } (g, \Box\text{I}) \\
3 & \quad\vert\quad\vert\ \underline{/\neg P/_y} & \text{A } (c, \neg\text{E}) \\
4 & \quad\vert\quad\vert\ /A/_u & \text{from } \Omega_{k^*} \cup \{s.y, /\neg P/_y\} \\
5 & \quad\vert\quad\vert\ \backslash\neg A\backslash_u & \text{from } \Omega_{k^*} \cup \{s.y, /\neg P/_y\} \\
6 & \quad\vert\ \backslash P\backslash_y & \text{3-5 } \neg\text{E} \\
7 & \backslash\Box P\backslash_s & \text{2-6 } \Box\text{I}
\end{array}$$

where, by construction, y is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NR4x} \backslash\Box P\backslash_s$; but $/\neg\Box P/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NR4x} /\neg\Box P/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

(v) (a) Suppose Ω_k is $\Omega_{k^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\}$. In this case, as above, Ω_{k*} is consistent and by construction, $/s.t.u/$,

$u.v.w \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there is some A_x such that $\Omega_{k^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\} \vdash^*_{NR4x} /A/_x$ and $\Omega_{k^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\} \vdash^*_{NR4x} \backslash\neg A\backslash_x$. So reason as follows,

1	Ω_{k^*}	
2	$/s.t.u/$	from Ω_{k^*}
3	$u.v.w$	from Ω_{k^*}
4	$t.v.y$	A (g, 2,3 AM^9_{10})
5	$/s.y.w/$	
6	$t.z.w$	A (g, 2,3 AM^9_{10})
7	$/s.v.z/$	
8	$\overline{\neg B}_0$	A (c, ¬E)
9	$\backslash\neg A\backslash_x$	from $\Omega_{k^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\}$
10	$/A/_x$	from $\Omega_{k^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\}$
11	B_0	8-10 ¬E
12	B_0	2,3,6-11 AM^9_{10}
13	B_0	2,3,4-12 AM^9_{10}

where, by construction, y and z are not in Ω_{k^*} are not 0. So $\Omega_{k^*} \vdash^*_{NR4x} B_0$; and similarly, $\Omega_{k^*} \vdash^*_{NR4x} \overline{\neg B}_0$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent. And similarly for (b).

(vi) Similar to (v).

(vii) Similar to (v).

(viii) Similar to (v).

For any i, Ω_i is consistent.

L9.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 and L6.6.

L9.7 If Γ_0 is consistent, then Γ' is a scapegoat set for →, □, C9/C10, C11, C12 and CM.

For →: Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NR4x} /\neg(A \to B)/_s$. By L9.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \not\vdash^*_{NR4x} \backslash\neg\neg(A \to B)\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg(A \to B)/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{\backslash s.yz\backslash, A_y, \overline{\neg B}_z\}$; so by construction, $\backslash s.y.z\backslash \in \Gamma'$, $A_y \in \Gamma'$ and $\overline{\neg B}_z \in \Gamma'$; so $\Gamma' \vdash^*_{NR4x} \backslash s.y.z\backslash$, $\Gamma' \vdash^*_{NR4x} A_y$ and $\Gamma' \vdash^*_{NR4x} \overline{\neg B}_z$. So Γ' is a scapegoat set for →.

For $\Box$: Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NR4x} /\neg\Box A/_s$. By L9.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NR4x} \backslash\neg\neg\Box A\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg\Box A/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.y, /\neg A/_y\}$; so by construction, $s.y \in \Gamma'$ and $/\neg A/_y \in \Gamma'$; so $\Gamma' \vdash^*_{NR4x} s.y$ and $\Gamma' \vdash^*_{NR4x} /\neg A/_y$. So Γ' is a scapegoat set for $\Box$.

For C9/C10: Suppose Γ_0 is consistent. (i) Suppose $\Gamma \vdash^*_{NR4x} /s.t.u/$ and $\Gamma \vdash^*_{NR4x} u.v.w$. By L9.6, Γ' is consistent; and by the constraints on subscripts, s, t, u, v, w are included in Γ'. Since $\Gamma \vdash^*_{NR4x} /s.t.u/$ and $\Gamma \vdash^*_{NR4x} u.v.w$, Γ' has just the same consequences as $\Gamma' \cup \{/s.t.u/, u.v.w\}$; so $\Gamma' \cup \{/s.t.u/, u.v.w\}$ is consistent, and for any Ω_j, $\Omega_j \cup \{/s.t.u/, u.v.w\}$ is consistent. So there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/s.t.u/, u.v.w\}$ and $\Omega_i = \Omega_{i^*} \cup \{t.v.y, /s.y.w/, t.z.w, /s.v.z/\}$; so by construction, $t.v.y$, $/s.y.w/$, $t.z.w$, $/s.v.z/ \in \Gamma'$; so there is some y such that $\Gamma \vdash^*_{NR4x} t.v.y$ and $\Gamma \vdash^*_{NR4x} /s.y.w/$, and there is some z such that $\Gamma \vdash^*_{NR4x} t.z.w$ and $\Gamma \vdash^*_{NR4x} /s.v.z/$. (ii) And similarly if $\Gamma \vdash^*_{NR4x} /s.u.t/$ and $\Gamma \vdash^*_{NR4x} \overline{u.v.w}$.

And similarly in the other cases.

C(I) We construct an interpretation $I = \langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ', except that if $\Gamma' \vdash^*_{NR4x} s \simeq t$ then $w_s = w_t$ (again, we might do this in the usual way by beginning with equivalence classes on subscripts). Then set $w_s \in /N/$ iff $\Gamma' \vdash^*_{NR4x} /n/[s]$; $\langle w_s, w_t\rangle \in M$ iff $\Gamma' \vdash^*_{NR4x} s.t$; $\langle w_s, w_t, w_u\rangle \in /R/$ iff $\Gamma' \vdash^*_{NR4x} /s.t.u/$; $\langle w_s, w_t\rangle \in \leq$ iff $\Gamma' \vdash^*_{NR4x} s \leq t$; $\langle w_s, w_t\rangle \in \leq^*$ iff $\Gamma' \vdash^*_{NR4x} s \leq^* t$; $\langle w_s, w_t\rangle \in \leq^\sharp$ iff $\Gamma' \vdash^*_{NR4x} s \leq^\sharp t$; and $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{NR4x} /p/_s$.

Note that the specification is consistent: Suppose $w_s = w_t$; then by construction, $\Gamma' \vdash^*_{NR4x} s \simeq t$; so by $\simeq$E, $\Gamma' \vdash^*_{NR4x} p_s$ iff $\Gamma' \vdash^*_{NR4x} p_t$ so $h_{w_s}(p) = h_{w_t}(p)$. And similarly in other cases.

L9.8 If Γ_0 is consistent then for $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ constructed as above, for any s included in Γ', $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NR4x} /A/_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L9.4, Γ' is s-maximal. By L9.6 and L9.7, Γ' is consistent and a scapegoat set for $\rightarrow$ and $\Box$. Now by induction on the number of operators in $/A/_s$,

Basis: If $/A/_s$ has no operators, then it is either $/n/[s]$ or a parameter $/p/_s$. But $h_{w_s}[/n/(s)] = 1$ iff $w_s \in /N/$; and by construction,

$w_s \in /N/$ iff $\Gamma' \vdash^*_{\mathit{NR4x}} /n/(s)$; so $h_{w_s}[/n/(s)] = 1$ iff $\Gamma' \vdash^*_{\mathit{NR4x}} /n/(s)$. And by construction, $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{\mathit{NR4x}} /p/_s$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{\mathit{NR4x}} /A/_s$.

Assp: For any i, $0 \leq i < k$, if $/A/_s$ has i operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{\mathit{NR4x}} /A/_s$.

Show: If $/A/_s$ has k operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{\mathit{NR4x}} /A/_s$.

If $/A/_s$ has k operators, then it is of the form $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$, or $\Box P$, where P and Q have $< k$ operators.

$(\neg)$

$(\wedge)$

$(\vee)$

$(\rightarrow)$ $/A/_s$ is $/P \rightarrow Q/_s$. (i) Suppose $h_{w_s}(/A/) = 1$ but $\Gamma' \nvdash^*_{\mathit{NR4x}} /A/_s$; then $h_{w_s}(/P \rightarrow Q/) = 1$, but $\Gamma' \nvdash^*_{\mathit{NR4x}} /P \rightarrow Q/_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{\mathit{NR4x}} \backslash\neg(P \rightarrow Q)\backslash_s$; but since Γ' is a scapegoat set for $\rightarrow$, there are some y, z such that $\Gamma' \vdash^*_{\mathit{NR4x}} /s.y.z/$, $\Gamma' \vdash^*_{\mathit{NR4x}} P_y$, and $\Gamma' \vdash^*_{\mathit{NR4x}} \overline{\neg Q}_z$; from the last of these, by consistency, $\Gamma' \nvdash^*_{\mathit{NR4x}} Q_z$; so by assumption, $h_{w_y}(P) = 1$ and $h_{w_z}(Q) = 0$; and since $\Gamma' \vdash^*_{\mathit{NR4x}} /s.y.z/$, by construction, $\langle w_s, w_y, w_z\rangle \in /R/$; so there are some y,$z \in W$ such that $s/R/yz$ and $h_{w_y}(P) = 1$ but $h_{w_z}(Q) = 0$; so by HR4x($\rightarrow$), $h_{w_s}(/P \rightarrow Q/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$ then $\Gamma' \vdash^*_{\mathit{NR4x}} /A/_s$.

(ii) Suppose $\Gamma' \vdash^*_{\mathit{NR4x}} /A/_s$ but $h_{w_s}(/A/) = 0$; then $\Gamma' \vdash^*_{\mathit{NR4x}} /P \rightarrow Q/_s$ but $h_{w_s}(/P \rightarrow Q/) = 0$. From the latter, by HR4x($\rightarrow$), there are some $t, u \in W$ such that $s/R/tu$ and either $h_{w_t}(P) = 1$ but $h_{w_u}(Q) = 0$ or $h_{w_u}(\overline{P}) = 1$ but $h_{w_t}(\overline{Q}) = 0$; so by construction, $\Gamma' \vdash^*_{\mathit{NR4x}} /s.t.u/$, and by assumption, either (a) $\Gamma' \vdash^*_{\mathit{NR4x}} P_t$ but $\Gamma' \nvdash^*_{\mathit{NR4x}} Q_u$ or (b) $\Gamma' \vdash^*_{\mathit{NR4x}} \overline{P}_u$ but $\Gamma' \nvdash^*_{\mathit{NR4x}} \overline{Q}_t$. Suppose (a); then $\Gamma' \vdash^*_{\mathit{NR4x}} P_t$ but $\Gamma' \nvdash^*_{\mathit{NR4x}} Q_u$. From the latter, by s-maximality, $\Gamma' \vdash^*_{\mathit{NR4x}} \overline{\neg Q}_u$; so $\Gamma' \vdash^*_{\mathit{NR4x}} /s.t.u/$, $\Gamma' \vdash^*_{\mathit{NR4x}} /P \rightarrow Q/_s$, and $\Gamma' \vdash^*_{\mathit{NR4x}} P_t$. So, by reasoning as follows,

1	$/s.t.u/$	from Γ'
2	$/P \rightarrow Q/_s$	from Γ'
3	P_t	from Γ'
4	Q_u	1-3 $\rightarrow$E

$\Gamma' \vdash^*_{\mathit{NR4x}} Q_u$; then Γ' is inconsistent. Suppose (b); then $\Gamma' \vdash^*_{\mathit{NR4x}} \overline{P}_u$ but $\Gamma' \nvdash^*_{\mathit{NR4x}} \overline{Q}_t$; from the latter, by s-maximality $\Gamma' \vdash^*_{\mathit{NR4x}}$

$\neg Q_t$; so $\Gamma' \vdash^*_{NR4x} /s.t.u/$, $\Gamma' \vdash^*_{NR4x} /P \to Q/_s$, and $\Gamma' \vdash^*_{NR4x} \overline{P}_u$. So, by reasoning as follows,

$$\begin{array}{r|ll} 1 & /s.t.u/ & \text{from } \Gamma' \\ 2 & /P \to Q/_s & \text{from } \Gamma' \\ 3 & \overline{P}_u & \text{from } \Gamma' \\ 4 & \overline{Q}_t & \text{1-3 } {\to}\text{E} \end{array}$$

so $\Gamma' \vdash^*_{NR4x} \overline{Q}_t$; so Γ' is inconsistent. In either case, then, Γ' is inconsistent. This is impossible; reject the assumption: if $\Gamma' \vdash^*_{NR4x} /A/_s$ then $h_{w_s}(/A/) = 1$.

So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NR4x} /A/_s$.

($\Box$) $/A/_s$ is $/\Box P/_s$. (i) Suppose $h_{w_s}(/A/) = 1$ but $\Gamma' \nvdash^*_{NR4x} /A/_s$; then $h_{w_s}(/\Box P/) = 1$, but $\Gamma' \nvdash^*_{NR4x} /\Box P/_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NR4x} \backslash\neg\Box P\backslash_s$; so, since Γ' is a scapegoat set for $\Box$, there is some y such that $\Gamma' \vdash^*_{NR4x} s.y$ and $\Gamma' \vdash^*_{NR4x} /\neg P/_y$; from the former of these, by construction, $\langle w_s, w_y\rangle \in M$; and from the latter, by consistency, $\Gamma' \nvdash^*_{NR4x} /P/_y$; so by assumption, $h_{w_y}(/P/) = 0$; but $w_s M w_y$; so by HR4x($\Box$), $h_{w_s}(/\Box P/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^*_{NR4x} /A/_s$.

(ii) Suppose $\Gamma' \vdash^*_{NR4x} /A/_s$ but $h_{w_s}(/A/) = 0$; then $\Gamma' \vdash^*_{NR4x} /\Box P/_s$ but $h_{w_s}(/\Box P/) = 0$. From the latter, by HR4x($\Box$), there is some $w_t \in W$ such that $w_s M w_t$ and $h_{w_t}(/P/) = 0$; so by assumption, $\Gamma' \nvdash^*_{NR4x} /P/_t$; but since $w_s M w_t$, by construction, $\Gamma' \vdash^*_{NR4x} s.t$; so by ($\Box$E), $\Gamma' \vdash^*_{NR4x} /P/_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NR4x} /A/_s$ then $h_{w_s}(/A/) = 1$.

So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NR4x} /A/_s$.

For any A_s, $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NR4x} /A/_s$.

L9.9 If Γ_0 is consistent, then $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ constructed as above is an $R4x$ interpretation.

For this, we need to show that the interpretation meets the constraints for NC and $\preceq$ along with C9/C10, C11, C12, CL and MC, CM, ρ, σ and τ.

Suppose Γ_0 is consistent. By L9.7, Γ' is a scapegoat set for C9/C10, C11, C12 and CM.

(NC) Suppose $w_s \in /N/$; then by construction $\Gamma' \vdash^*_{NR4x} /n/[s]$. (i) Suppose $w_s/R/w_t w_u$; then by construction $\Gamma' \vdash^*_{NR4x} /s.t.u/$; so by NE, $\Gamma' \vdash^*_{NR4x} t \simeq u$; so by construction $w_t = w_u$. (ii) Suppose $w_t = w_u$; then by construction, $\Gamma' \vdash^*_{NR4x} t \simeq u$; so by NE, $\Gamma' \vdash^*_{NR4x} /s.t.u/$; so by construction, $w_s/R/w_t w_u$.

($\leq$) Suppose $\langle w_s, w_t\rangle \in \leq$; then $\Gamma' \vdash^*_{NR4x} s \leq t$. (i) Suppose $h_{w_s}(p) = 1$; then by construction, $\Gamma' \vdash^*_{NR4x} p_s$; so by $\leq$E, $\Gamma' \vdash^*_{NR4x} p_t$; so by construction $h_{w_t}(p) = 1$. Suppose $h_{w_t}(\overline{p}) = 1$; then by construction $\Gamma' \vdash^*_{NR4x} \overline{p}_t$; then,

$$\begin{array}{r|lll}
1 & s \leq t & & \text{from } \Gamma' \\
2 & \overline{p}_t & & \text{from } \Gamma' \\
3 & \quad | \; \underline{\neg p_s} & & \text{A } (g\ \neg\text{E}) \\
4 & \quad | \; \neg p_t & & 1,3 \leq\text{E} \\
5 & \quad | \; \overline{p}_t & & 2\ \text{R} \\
6 & \overline{p}_s & & 3\text{-}5\ \neg\text{E}
\end{array}$$

$\Gamma' \vdash^*_{NR4x} \overline{p}_s$; so by construction, $h_{w_s}(\overline{p}) = 1$. (ii) Suppose $w_t R w_x w_y$ and $w_s \notin N$; then by construction $\Gamma' \vdash^*_{NR4x} t.x.y$ and $\Gamma' \nvdash^*_{NR4x} n(s)$ so that by s-maximality (with the washed out overline for this expression) $\Gamma' \vdash^*_{NR4x} \neg n(s)$; so by $\leq_R$E, $\Gamma' \vdash^*_{NR4x} s.x.y$; so by construction, $w_s R w_x w_y$. Suppose $w_t R w_x w_y$ and $w_s \in N$; then by construction $\Gamma' \vdash^*_{NR4x} t.x.y$ and $\Gamma' \vdash^*_{NR4x} n(s)$; so by $\leq_R$E, $\Gamma' \vdash^*_{NR4x} x \leq y$; so by construction, $\langle w_x, w_y\rangle \in \leq$. (iii) Suppose $w_s \overline{R} w_x w_y$ and $w_t \notin \overline{N}$; then by construction $\Gamma' \vdash^*_{NR4x} \overline{s.x.y}$ and $\Gamma' \nvdash^*_{NR4x} \overline{n}(t)$ so that by s-maximality, $\Gamma' \vdash^*_{NR4x} \neg\overline{n}(t)$; so by $\leq_R$E, $\Gamma' \vdash^*_{NR4x} \overline{t.x.y}$; so by construction, $w_t \overline{R} w_x w_y$. Suppose $w_s \overline{R} w_x w_y$ and $w_t \in \overline{N}$; then by construction $\Gamma' \vdash^*_{NR4x} \overline{s.x.y}$ and $\Gamma' \vdash^*_{NR4x} \overline{n}(t)$; so by $\leq_R$E, $\Gamma' \vdash^*_{NR4x} x \leq y$; so by construction, $\langle w_x, w_y\rangle \in \leq$.

($\leq^*$)

($\leq^\sharp$)

(C9/C10) (i) Suppose there is a w_u such that $\langle w_s, w_t, w_u\rangle \in /R/$ and $\langle w_u, w_v, w_w\rangle \in R$; then by construction, $\Gamma' \vdash^*_{NR4x} /s.t.u/$ and $\Gamma' \vdash^*_{NR4x} u.v.w$; so, since Γ' is a C9/C10 scapegoat set, there is a y such that $\Gamma' \vdash^*_{NR4x} t.v.y$ and $\Gamma' \vdash^*_{NR4x} /s.y.w/$, and there is a z such that $\Gamma' \vdash^*_{NR4x} t.z.w$ and $\Gamma' \vdash^*_{NR4x} /s.v.z/$; so by construction, $\langle w_t, w_v, w_y\rangle \in R$, $\langle w_s, w_y, w_w\rangle \in /R/$, $\langle w_t, w_z, w_w\rangle \in R$ and $\langle w_s, w_v, w_z\rangle \in /R/$. (ii) Suppose there is a w_u such that $\langle w_s, w_u, w_t\rangle \in /R/$ and $\langle w_u, w_v, w_w\rangle \in \overline{R}$; then by construction, $\Gamma' \vdash^*_{NR4x} /s.u.t/$ and $\Gamma' \vdash^*_{NR4x} \overline{u.v.w}$; so, since Γ' is a C9/C10 scapegoat set, there is a y such that $\Gamma' \vdash^*_{NR4x} \overline{t.v.y}$

and $\Gamma' \vdash^{*}_{NR4x} /s.y.w/$, and there is a z such that $\Gamma' \vdash^{*}_{NR4x} \overline{t.z.w}$ and $\Gamma' \vdash^{*}_{NR4x} /s.v.z/$; so by construction, $\langle w_t, w_v, w_y\rangle \in \overline{R}$, $\langle w_s, w_y, w_w\rangle \in /R/$, $\langle w_t, w_z, w_w\rangle \in \overline{R}$ and $\langle w_s, w_v, w_z\rangle \in /R/$. So C9 and C10 are satisfied.

(C11)

(C12)

(CL) (i) Suppose $w_s \in /N/$; then by construction $\Gamma' \vdash^{*}_{NR4x} /n/(s)$; so by CL, $\Gamma' \vdash^{*}_{NR4x} \backslash n\backslash(s)$; so by construction, $w_s \in \backslash N\backslash$. (ii) Suppose $w_s \in N$; then by construction, $\Gamma' \vdash^{*}_{NR4x} /n/(s)$. Suppose $h_{w_s}(/p/) = 1$; then by construction $\Gamma' \vdash^{*}_{NR4x} /p/_s$; so by CL, $\Gamma' \vdash^{*}_{NR4x} \backslash p\backslash_s$; so by construction $h_{w_s}(\backslash p\backslash) = 1$. Suppose $h_{w_s}(/p/) = 0$; then by construction $\Gamma' \nvdash^{*}_{NR4x} /p/_s$; so by s-maximality, $\Gamma' \vdash^{*}_{NR4x} \backslash\neg p\backslash_s$; so by CL, $\Gamma' \vdash^{*}_{NR4x} /\neg p/_s$; and with consistency, $\Gamma' \nvdash^{*}_{NR4x} \backslash p\backslash_s$; and by construction, $h_{w_s}(\backslash p\backslash) = 0$.

(MC)

(CM) (i) Suppose $w_s \preceq w_t$ and $w_t M w_u$; then by construction, $\Gamma' \vdash^{*}_{NR4x} s \preceq t$ and $\Gamma' \vdash^{*}_{NR4x} t.u$; so since Γ' is a scapegoat set for CM, there is a y such that $\Gamma' \vdash^{*}_{NR4x} y \preceq u$ and $\Gamma' \vdash^{*}_{NR4x} s.y$; so by construction $\langle w_y, w_u\rangle \in \preceq$ and $\langle w_s, w_y\rangle \in M$. (ii) Suppose $w_s \preceq w_t$ and $w_s M w_u$; then by construction, $\Gamma' \vdash^{*}_{NR4x} s \preceq t$ and $\Gamma' \vdash^{*}_{NR4x} s.u$; so since Γ' is a scapegoat set for CM, there is a y such that $\Gamma' \vdash^{*}_{NR4x} u \preceq y$ and $\Gamma' \vdash^{*}_{NR4x} t.y$; so by construction $\langle w_u, w_y\rangle \in \preceq$ and $\langle w_t, w_y\rangle \in M$.

(ρ)

(σ)

(τ)

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L9.10 If Γ_0 is consistent, then $h_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 and L6.9.

Main result: Suppose $\Gamma \models_{R4x} A$ but $\Gamma \nvdash_{NR4x} A$. Then $\Gamma_0 \models^{*}_{R4x} A_0$ but $\Gamma_0 \nvdash^{*}_{NR4x} A_0$. By (DN), if $\Gamma_0 \vdash^{*}_{NR4x} \neg\neg A_0$, then $\Gamma_0 \vdash^{*}_{NR4x} A_0$; so $\Gamma_0 \nvdash^{*}_{NR4x} \neg\neg A_0$; so by L9.2, $\Gamma_0 \cup \{\overline{\neg A_0}\}$ is consistent; so by L9.9 and L9.10, there is an $R4x$ interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, \preceq, h\rangle$ constructed as above such that $h_m(\Gamma_0 \cup \{\overline{\neg A_0}\}) = 1$; so $h_{m(0)}(\overline{\neg A}) = 1$; so by HR4($\neg$), $h_{m(0)}(A) = 0$; so $h_m(\Gamma_0) = 1$ and $h_{m(0)}(A) = 0$; so by VR4x*, $\Gamma_0 \not\models^{*}_{R4x} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{R4x} A$, then $\Gamma \vdash_{NR4x} A$.

10 Many-Valued Modal Logics: $\boldsymbol{K}_{Lx}$ (appendix)

This section is developed again in terms as for section 7. This smooths presentation, and applies to Priest as before.

10.1 Language / Semantic Notions

LK$_{\mathrm{L}}$ The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators $\neg$, $\wedge$, $\vee$, $\Box$, and $\Diamond$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $\Box A$, and $\Diamond A$. $A \supset B$ abbreviates $\neg A \vee B$. If A is a formula so formed, so is $\overline{A}$.

Let $/A/$ and $\backslash A\backslash$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $/A/$ and $\backslash A\backslash$ are opposite.

IK$_{\mathrm{L}}$ An INTERPRETATION is $\langle W, R, h\rangle$ where W is a set of worlds; $R \subseteq R^2$ is a modal access relation; and $h_w(/p/) = 0$ or $h_w(/p/) = 1$. Optionally interpretations are subject to,

exc for no p are both $h_w(p) = 1$ and $h_w(\overline{p}) = 0$

exh for any p either $h_w(p) = 1$ or $h_w(\overline{p}) = 0$

ρ Reflexivity: for all x, xMx.

σ Symmetry: for all x, y if xMy then yMx.

τ Transitivity: for all x, y, z if xMy and yMz then xMz.

We get K_{LP} with *exh*, and K_{K3} with *exc*. K_{FDE} has neither of these constraints. We recover classical K with both. These logics may add ρ, σ and τ in the natural way.

HK$_{\mathrm{L}}$ For complex expressions,

($\neg$) $h_w(/\neg P/) = 1$ iff $h_w(\backslash P\backslash) = 0$

($\wedge$) $h_w(/P \wedge Q/) = 1$ iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$

($\vee$) $h_w(/P \vee Q/) = 1$ iff $h_w(/P/) = 1$ or $h_w(/Q/) = 1$

($\Box$) $h_w(/\Box P/) = 1$ iff there is no $x \in W$ such that wMx and $h_x(/P/) = 0$

($\Diamond$) $h_w(/\Diamond P/) = 1$ iff there is some $x \in W$ such that wMx and $h_x(/P/) = 1$

For a set Γ of formulas, $h_w(\Gamma) = 1$ iff $h_w(/P/) = 1$ for each $/P/ \in \Gamma$; then,

VK$_\text{L}$ $\Gamma \models_{K_{Lx}} P$ iff there is no K_{Lx} interpretation $\langle W, R, h\rangle$ and w such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

10.2 Natural Derivations: $\boldsymbol{NK_L}$

Derivations combine methods from modal and multi-valued logics in the natural way. Allow subscripts to indicate worlds. (D) corresponds to *exc* and (U) to *exh*.

$$\textbf{D}\ \begin{array}{|l} P_s \\ \\ \overline{P}_s \end{array} \qquad \textbf{R}\ \begin{array}{|l} /P/_s \\ \\ /P/_s \end{array} \qquad \neg\textbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{/P/_s} \\ \\ /\!\!/Q/\!\!/_t \\ \backslash\!\!\backslash\neg Q\backslash\!\!\backslash_t \end{array} \\ \backslash\neg P\backslash_s \end{array} \qquad \neg\textbf{E}\ \begin{array}{|l} \begin{array}{|l} \underline{/\neg P/_s} \\ \\ /\!\!/Q/\!\!/_t \\ \backslash\!\!\backslash\neg Q\backslash\!\!\backslash_t \end{array} \\ \backslash P\backslash_s \end{array}$$

$$\textbf{U}\ \begin{array}{|l} \overline{P}_s \\ \\ P_s \end{array} \qquad \wedge\textbf{I}\ \begin{array}{|l} /P/_s \\ /Q/_s \\ \\ /P \wedge Q/_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} /P \wedge Q/_s \\ \\ /P/_s \end{array} \qquad \wedge\textbf{E}\ \begin{array}{|l} /P \wedge Q/_s \\ \\ /Q/_s \end{array}$$

$$\vee\textbf{I}\ \begin{array}{|l} /P/_s \\ \\ /P \vee Q/_s \end{array} \qquad \vee\textbf{I}\ \begin{array}{|l} /P/_s \\ \\ /Q \vee P/_s \end{array} \qquad \vee\textbf{E}\ \begin{array}{|l} /P \vee Q/_s \\ \begin{array}{|l} \underline{/P/_s} \\ \\ /\!\!/R/\!\!/_t \end{array} \\ \begin{array}{|l} \underline{/Q/_s} \\ \\ /\!\!/R/\!\!/_t \end{array} \\ /\!\!/R/\!\!/_t \end{array}$$

$$\supset\textbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{/P/_s} \\ \\ \backslash Q\backslash_s \end{array} \\ \backslash P \supset Q\backslash_s \end{array} \qquad \supset\textbf{E}\ \begin{array}{|l} \backslash P \supset Q\backslash_s \\ /P/_s \\ \\ \backslash Q\backslash_s \end{array}$$

$$\textbf{AM}\rho\ \begin{array}{|l} \\ \\ \\ s.s \end{array} \qquad \textbf{AM}\sigma\ \begin{array}{|l} s.t \\ \\ \\ t.s \end{array} \qquad \textbf{AM}\tau\ \begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

$$\Box\textbf{I}\ \begin{array}{|l} \begin{array}{|l} \underline{s.t} \\ \\ /P/_t \end{array} \\ /\Box P/_s \end{array} \qquad \Box\textbf{E}\ \begin{array}{|l} /\Box P/_s \\ s.t \\ \\ /P/_t \end{array} \qquad \Diamond\textbf{I}\ \begin{array}{|l} /P/_t \\ s.t \\ \\ /\Diamond P/_s \end{array} \qquad \Diamond\textbf{E}\ \begin{array}{|l} /\Diamond P/_s \\ \begin{array}{|l} s.t \\ \underline{/P/_t} \\ \\ /\!\!/Q/\!\!/_u \end{array} \\ /\!\!/Q/\!\!/_u \end{array}$$

($\Box$I) where t does not appear in any undischarged premise or assumption

($\Diamond$E) where t does not appear in any undischarged premise or assumption and is not u

Every subscript is 0, appears in a premise, or in the t place of an assumption for $\Box$I or $\Diamond$E. Where the members of Γ and A are without overlines or subscripts, let Γ_0 be the members of Γ, each with subscript 0. Then,

NK$_{\text{L}}$ $\Gamma \vdash_{NK_{Lx}} A$ iff there is an NK_{Lx} derivation of A_0 from Γ_0.

We allow standard two-way derived rules (including MN) with overlines and subscripts constant throughout. MT, NB and DS appear in the forms,

MT	$/P \supset Q/_s$	**NB**	$/P \equiv Q/_s$	$/P \equiv Q/_s$	**DS**	$/P \lor Q/_s$	$/P \lor Q/_s$
	$\backslash\neg Q\backslash_s$		$\backslash\neg P\backslash_s$	$\backslash\neg Q\backslash_s$		$\backslash\neg P\backslash_s$	$\backslash\neg Q\backslash_s$
	$/\neg P/_s$		$/\neg Q/_s$	$/\neg P/_s$		$/Q/_s$	$/P/_s$

Examples. The first couple cases are matched to show an equivalent result by different means.

$\Box A \land \neg\Box B \vdash_{NK_{FDE}} \Diamond(A \land \neg B)$

1	$(\Box A \land \neg B)_0$	P
2	$\Box A_0$	1 $\land$E
3	$\neg\Box B_0$	1 $\land$E
4	$\Diamond\neg B_0$	3 MN
5	$\mid 0.1$	A (g 4$\Diamond$E)
6	$\mid \neg B_1$	
7	$\mid A_1$	2,5 $\Box$E
8	$\mid (A \land \neg B)_1$	7,6 $\land$I
9	$\mid \Diamond(A \land \neg B)_0$	5,8 $\Diamond$I
10	$\Diamond(A \land \neg B)_0$	4,5-9 $\Diamond$E

$\Box A \land \neg\Box B \vdash_{NK_{FDE}} \neg\Box\neg(A \land \neg B)$

1	$(\Box A \land \neg\Box B)_0$	P
2	$\Box A_0$	1 $\land$E
3	$\mid \overline{\Box\neg(A \land \neg B)}_0$	A (c, $\neg$I)
4	$\mid \mid 0.1$	A (g, $\Box$I)
5	$\mid \mid \mid \neg B_1$	A (c, $\neg$E)
6	$\mid \mid \mid A_1$	2,4 $\Box$E
7	$\mid \mid \mid (A \land \neg B)_1$	6,5 $\land$I
8	$\mid \mid \mid \overline{\neg(A \land \neg B)}$	3 $\Box$E
9	$\mid \mid \overline{B}_1$	5-8 $\neg$E
10	$\mid \overline{\Box B}_0$	5-9 $\Box$I
11	$\mid \neg\Box B_0$	1 $\land$E
12	$\neg\Box\neg(A \land \neg B)_0$	3-11 $\neg$I

$\Box A \vdash_{NK_{LP\tau}} \Box\Box A$

1	$\vert\ \vert\ \overline{\Box A}_0$	A (g, $\supset$I)
2	$\vert\ \vert\ \vert\ 0.1$	A (g, $\Box$I)
3	$\vert\ \vert\ \vert\ \vert\ 1.2$	A (g, $\Box$I)
4	$\vert\ \vert\ \vert\ \vert\ 0.2$	2,3 AMτ
5	$\vert\ \vert\ \vert\ \vert\ \overline{A}_2$	1,4 $\Box$E
6	$\vert\ \vert\ \vert\ \vert\ A_2$	5 U
7	$\vert\ \vert\ \vert\ \Box A_1$	3-6 $\Box$I
8	$\vert\ \vert\ \Box\Box A_0$	2-7 $\Box$I
9	$\vert\ (\Box A \supset \Box\Box A)_0$	1-8 $\supset$I

$\Box(\Diamond A \supset B) \vdash_{NK_{FDE\sigma\tau}} \Box(A \supset \Box B)$

1	$\vert\ \Box(\Diamond A \supset B)_0$	P
2	$\vert\ \vert\ 0.1$	A (g, $\Box$I)
3	$\vert\ \vert\ \vert\ \overline{A}_1$	A (g, $\supset$I)
4	$\vert\ \vert\ \vert\ \vert\ 1.2$	A (g, $\Box$I)
5	$\vert\ \vert\ \vert\ \vert\ 2.1$	4 AMσ
6	$\vert\ \vert\ \vert\ \vert\ \overline{\Diamond A}_2$	3,5 $\Diamond$I
7	$\vert\ \vert\ \vert\ \vert\ 0.2$	2,5 AMτ
8	$\vert\ \vert\ \vert\ \vert\ (\Diamond A \supset B)_2$	1,7 $\Box$E
9	$\vert\ \vert\ \vert\ \vert\ B_2$	8,6 $\supset$E
10	$\vert\ \vert\ \vert\ \Box B_1$	4-9 $\Box$I
11	$\vert\ \vert\ (A \supset \Box B)_1$	3-10 $\supset$I
12	$\vert\ \Box(A \supset \Box B)_0$	2-11 $\Box$I

10.3 Soundness and Completeness

Preliminaries: Begin with generalized notions of validity. For a model $\langle W, R, h\rangle$, let m be a map from subscripts into W. Then say $\langle W, R, h\rangle_m$ is $\langle W, R, h\rangle$ *with* map m. Then, where Γ is a set of expressions of our language for derivations, $h_m(\Gamma) = 1$ iff for each $/A_s/ \in \Gamma$, $h_{m(s)}(/A/) = 1$, and for each $s.t \in \Gamma$, $\langle m(s), m(t)\rangle \in R$. Now expand notions of validity for subscripts, overlines, and alternate expressions as indicated in double brackets as follows,

VK$_{\text{L}}$* $\Gamma \models^*_{K_{Lx}} /A/_s$ $\llbracket s.t \rrbracket$ iff there is no K_{Lx} interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma) = 1$ but $h_{m(s)}(/A/) = 0$ $\llbracket \langle m(s), m(t)\rangle \notin R \rrbracket$.

NK$_{\text{L}}$* $\Gamma \vdash_{NK_{Lx}} *]/A/_s$ $\llbracket s.t \rrbracket$ iff there is an NK_{Lx} derivation of $/A/_s$ $\llbracket s.t \rrbracket$ from the members of Γ.

These notions reduce to the standard ones when all the members of Γ and A are without overlines and have subscript 0 (and so do not include expressions of the sort $s.t$). As usual, for the following, cases omitted are like ones worked, and so left to the reader.

Theorem 10.1 *NK_{L} is sound: If $\Gamma \vdash_{NK_{L\alpha}} A$ then $\Gamma \vDash_{K_{L\alpha}} A$.*

L10.1 If $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{K_{L\alpha}} /P/_s \llbracket s.t \rrbracket$ then $\Gamma' \vDash^*_{K_{L\alpha}} /P/_s \llbracket s.t \rrbracket$.

Suppose $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{K_{L\alpha}} /P/_s \llbracket s.t \rrbracket$, but $\Gamma' \nvDash^*_{K_{L\alpha}} /P/_s \llbracket s.t \rrbracket$. From the latter, by $\mathrm{VK_L}^*$, there is some $K_{L\alpha}$ interpretation $\langle W, R, h \rangle_m$ such that $h_m(\Gamma') = 1$ but $h_{m(s)}(/P/) = 0$ $\llbracket \langle m(s), m(t) \rangle \notin R \rrbracket$. But since $h_m(\Gamma') = 1$ and $\Gamma \subseteq \Gamma'$, $h_m(\Gamma) = 1$; so $h_m(\Gamma) = 1$ but $h_{m(s)}(/P/) = 0$ $\llbracket \langle m(s), m(t) \rangle \notin R \rrbracket$; so by $\mathrm{VK_L}^*$, $\Gamma \nvDash^*_{K_{L\alpha}} /P/_s \llbracket s.t \rrbracket$. This is impossible; reject the assumption: if $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash^*_{K_{L\alpha}} /P/_s \llbracket s.t \rrbracket$, then $\Gamma' \vDash^*_{K_{L\alpha}} /P/_s \llbracket s.t \rrbracket$.

Main result: For each line in a derivation let $\mathcal{P}_i$ be the expression on line i and Γ_i be the set of all premises and assumptions whose scope includes line i. We set out to show "generalized" soundness: if $\Gamma \vdash^*_{NK_{L\alpha}} \mathcal{P}$ then $\Gamma \vDash^*_{K_{L\alpha}} \mathcal{P}$. As above, this reduces to the standard result when $\mathcal{P}$ and all the members of Γ are without overlines and have subscript 0. Suppose $\Gamma \vdash^*_{NK_{L\alpha}} \mathcal{P}$. Then there is a derivation of $\mathcal{P}$ from premises in Γ where $\mathcal{P}$ appears under the scope of the premises alone. By induction on line number of this derivation, we show that for each line i of this derivation, $\Gamma_i \vDash^*_{K_{L\alpha}} \mathcal{P}_i$. The case when $\mathcal{P}_i = \mathcal{P}$ is the desired result.

Basis: $\mathcal{P}_1$ is a premise or an assumption $/A/_s \llbracket s.t \rrbracket$. Then $\Gamma_1 = \{/A/_s\}$ $\llbracket \{s.t\} \rrbracket$; so for any $\langle W, R, h \rangle_m$, $h_m(\Gamma_1) = 1$ iff $h_{m(s)}(/A/) = 1$ $\llbracket \langle m(s), m(t) \rangle \in R \rrbracket$; so there is no $\langle W, R, h \rangle_m$ such that $h_m(\Gamma_1) = 1$ but $h_{m(s)}(/A/) = 0$ $\llbracket \langle m(s), m(t) \rangle \notin R \rrbracket$. So by $\mathrm{VK_L}^*$, $\Gamma_1 \vDash^*_{K_{L\alpha}} /A/_s \llbracket s.t \rrbracket$, where this is just to say, $\Gamma_1 \vDash^*_{K_{L\alpha}} \mathcal{P}_1$.

Assp: For any $i, 1 \leq i < k, \Gamma_i \vDash^*_{K_{L\alpha}} \mathcal{P}_i$.

Show: $\Gamma_k \vDash^*_{K_{L\alpha}} \mathcal{P}_k$.

$\mathcal{P}_k$ is either a premise, an assumption, or arises from previous lines by R, $\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\neg$I, $\neg$E, $\Box$I, $\Box$E, $\Diamond$I, $\Diamond$E or, depending on the system, AMρ, AMσ, AMτ, D, or U. If $\mathcal{P}_k$ is a premise or an assumption, then as in the basis, $\Gamma_k \vDash^*_{K_{L\alpha}} \mathcal{P}_k$. So suppose $\mathcal{P}_k$ arises by one of the rules.

(R)

($\wedge$I)

($\wedge$E)

($\vee$I)

($\vee$E)

($\neg$I)

($\neg$E)

($\Box$I) If $\mathcal{P}_k$ arises by $\Box$I, then the picture is like this,

$$\begin{array}{r|l|l}
 & & s.t \\ \cline{3-3}
 & & \\
j & & /A/_t \\
k & \multicolumn{2}{l}{/\Box A/_s}
\end{array}$$

where $j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption), and $\mathcal{P}_k$ is $/\Box A/_s$. By assumption, $\Gamma_j \models^*_{K_{Lx}} /A/_t$; but by the nature of access, $\Gamma_j \subseteq \Gamma_k \cup \{s.t\}$; so by L10.1, $\Gamma_k \cup \{s.t\} \models^*_{K_{Lx}} /A/_t$. Suppose $\Gamma_k \not\models^*_{K_{Lx}} /\Box A/_s$; then by VK$_{\mathrm{L}}$*, there is some K_{Lx} interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(/\Box A/) = 0$; so by HK$_{\mathrm{L}}$($\Box$), there is some $w \in W$ such that $m(s)Rw$ and $h_w(/A/) = 0$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, h\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(t) = w$ and $m'(s) = m(s)$, $\langle m'(s), m'(t)\rangle \in R$; so $h_{m'}(\Gamma_k \cup \{s.t\}) = 1$; so by VK$_{\mathrm{L}}$*, $h_{m'(t)}(/A/) = 1$. But $m'(t) = w$; so $h_w(/A/) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_{Lx}} /\Box A/_s$, which is to say, $\Gamma_k \models^*_{K_{Lx}} \mathcal{P}_k$.

($\Box$E) If $\mathcal{P}_k$ arises by $\Box$E, then the picture is like this,

$$\begin{array}{r|l}
i & /\Box A/_s \\
j & s.t \\
 & \\
k & /A/_t
\end{array}$$

where $i, j < k$ and $\mathcal{P}_k$ is $/A/_t$. By assumption, $\Gamma_i \models^*_{K_{Lx}} /\Box A/_s$ and $\Gamma_j \models^*_{K_{Lx}} s.t$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k$; so by L10.1, $\Gamma_k \models^*_{K_{Lx}} /\Box A/_s$ and $\Gamma_k \models^*_{K_{Lx}} s.t$. Suppose $\Gamma_k \not\models^*_{K_{Lx}} /A/_t$; then by VK$_{\mathrm{L}}$*, there is some K_{Lx} interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(t)}(/A/) = 0$; since $h_m(\Gamma_k) = 1$, by VK$_\mathrm{L}$*, $h_{m(s)}(/\Box A/) = 1$ and $\langle m(s), m(t)\rangle \in R$; from the first of these, by HK$_\mathrm{L}$($\Box$), any w such that $m(s)Rw$ has $h_w(/A/) = 1$; so $h_{m(t)}(/A/) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{K_{Lx}} /A/_t$, which is to say, $\Gamma_k \vDash^*_{K_{Lx}} \mathcal{P}_k$.

($\Diamond$I)

($\Diamond$E) If $\mathcal{P}_k$ arises by $\Diamond$E, then the picture is like this,

$$
\begin{array}{r|l}
i & /\Diamond A/_s \\
 & \quad\vert\; s.t \\
 & \quad\vert\; \underline{/A/_t} \\
 & \quad\vert \\
 & \quad\vert \\
j & \quad\vert\; /\!/B/\!/_u \\
k & /\!/B/\!/_u
\end{array}
$$

where $i, j < k$, t does not appear in any member of Γ_k (in any undischarged premise or assumption) and is not u, and $\mathcal{P}_k$ is $/\!/B/\!/_u$. By assumption, $\Gamma_i \vDash^*_{K_{Lx}} /\Diamond A/_s$ and $\Gamma_j \vDash^*_{K_{Lx}} /\!/B/\!/_u$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$ and $\Gamma_j \subseteq \Gamma_k \cup \{s.t, /A/_t\}$; so by L10.1, $\Gamma_k \vDash^*_{K_{Lx}} /\Diamond A/_s$ and $\Gamma_k \cup \{s.t, /A/_t\} \vDash^*_{K_{Lx}} /\!/B/\!/_u$. Suppose $\Gamma_k \nvDash^*_{K_{Lx}} /\!/B/\!/_u$; then by VK$_\mathrm{L}$*, there is some K_{Lx} interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(u)}(/\!/B/\!/) = 0$; since $h_m(\Gamma_k) = 1$, by VK$_\mathrm{L}$*, $h_{m(s)}(/\Diamond A/) = 1$; so by HK$_\mathrm{L}$($\Diamond$), there is some $w \in W$ such that $m(s)Rw$ and $h_w(/A/) = 1$. Now consider a map m' like m except that $m'(t) = w$, and consider $\langle W, R, h\rangle_{m'}$; since t does not appear in Γ_k, it remains that $h_{m'}(\Gamma_k) = 1$; and since $m'(s) = m(s)$ and $m'(t) = w$, $h_{m'(t)}(/A/) = 1$ and $\langle m'(s), m'(t)\rangle \in R$; so $h_{m'}(\Gamma_k \cup \{s.t, /A/_t\}) = 1$; so by VK$_\mathrm{L}$*, $h_{m'(u)}(/\!/B/\!/) = 1$. But since $t \neq u$, $m'(u) = m(u)$; so $h_{m(u)}(/\!/B/\!/) = 1$. This is impossible; reject the assumption: $\Gamma_k \vDash^*_{K_{Lx}} /\!/B/\!/_u$, which is to say, $\Gamma_k \vDash^*_{K_{Lx}} \mathcal{P}_k$.

(AMρ)

(AMσ)

(AMτ)

(D) If $\mathcal{P}_k$ arises by D, then the picture is like this,

$$
\begin{array}{r|l}
i & A_s \\
 & \\
k & \overline{A}_s
\end{array}
$$

where $i < k$ and $\mathcal{P}_k$ is $\overline{A}_s$. Where this rule is included in NK_{L}, K_{Lx} has condition *exc*, so no interpretation has $h_x(p) = \{1, 0\}$. By assumption, $\Gamma_i \models^*_{K_{Lx}} A_s$; but by the nature of access, $\Gamma_i \subseteq \Gamma_k$; so by L10.1*a*, $\Gamma_k \models^*_{K_{Lx}} A_s$. Suppose $\Gamma_k \not\models^*_{K_{Lx}} \overline{A}_s$; then by $\mathrm{VK}_{\mathrm{L}}{}^*$, there is some K_{Lx} interpretation $\langle W, R, h\rangle_m$ such that $h_m(\Gamma_k) = 1$ but $h_{m(s)}(\overline{A}) = 0$; since $h(\Gamma_k) = 1$, by $\mathrm{VK}_{\mathrm{L}}{}^*$, $h_{m(s)}(A) = 1$. But for these interpretations, for any A and any $x \in W$, if $h_x(A) = 1$ then $h_x(\overline{A}) = 1$.

Basis: A is a parameter p. Suppose $h_x(A) = 1$; then $h_x(p) = 1$; so $1 \in h_x(p)$; so by *exc*, $0 \notin h_x(p)$; so $h_x(\overline{p}) = 1$; so $h_x(\overline{A}) = 1$.

Assp: For any i, $0 \leq i < k$, if A has i operators, and $h_x(A) = 1$, then $h_x(\overline{A}) = 1$.

Show: If A has k operators, and $h_x(A) = 1$, then $h_x(\overline{A}) = 1$.

If A has k operators, then A is of the form, $\neg P$, $P \wedge Q$, $P \vee Q$, $\Box P$, or $\Diamond P$, where P and Q have $< k$ operators.

($\neg$) A is $\neg P$. Suppose $h_x(A) = 1$; then $h_x(\neg P) = 1$; so by $\mathrm{HK}_{\mathrm{L}}(\neg)$, $h_x(\overline{P}) = 0$; so by assumption, $h_x(P) = 0$; so by $\mathrm{HK}_{\mathrm{L}}(\neg)$, $h_x(\overline{\neg P}) = 1$, which is to say, $h_x(\overline{A}) = 1$.

($\wedge$) A is $P \wedge Q$. Suppose $h_x(A) = 1$; then $h_x(P \wedge Q) = 1$; so by $\mathrm{HK}_{\mathrm{L}}(\wedge)$, $h_x(P) = 1$ and $h_x(Q) = 1$; so by assumption, $h_x(\overline{P}) = 1$ and $h_x(\overline{Q}) = 1$; so by $\mathrm{HK}_{\mathrm{L}}(\wedge)$, $h_x(\overline{P \wedge Q}) = 1$, which is to say $h_x(\overline{A}) = 1$.

($\vee$)

($\Box$) A is $\Box P$. Suppose $h_x(A) = 1$; then $h_x(\Box P) = 1$; so by $\mathrm{HK}_{\mathrm{L}}(\Box)$, any $w \in W$ such that xRw has $h_w(P) = 1$; so by assumption, any $w \in W$ such that xRw has $h_w(\overline{P}) = 1$; so by $\mathrm{HK}_{\mathrm{L}}(\Box)$, $h_x(\overline{\Box P}) = 1$, which is to say, $h_x(\overline{A}) = 1$.

($\Diamond$)

For any A and any $x \in W$, if $h_x(A) = 1$, then $h_x(\overline{A}) = 1$.

So, returning to the case for (D), $h_{m(s)}(\overline{A}) = 1$. This is impossible; reject the assumption: $\Gamma_k \models^*_{K_{Lx}} \overline{A}$, which is to say, $\Gamma_k \models^*_{K_{Lx}} \mathcal{P}_k$.

(U)

For any i, $\Gamma_i \models^*_{K_{Lx}} \mathcal{P}_i$.

THEOREM 10.2 *NK_{L} is complete: if $\Gamma \models_{K_{Lx}} A$ then $\Gamma \vdash_{NK_{Lx}} A$.*

Suppose $\Gamma \models_{K_{Lx}} A$; then $\Gamma_0 \models^*_{K_{Lx}} A_0$; we show that $\Gamma_0 \vdash^*_{NK_{Lx}} A_0$. As usual, this reduces to the standard notion. For the following, fix on some particular K_{Lx}. Then definitions of *consistency* etc. are relative to it.

CON Γ is CONSISTENT iff there is no A_s such that $\Gamma \vdash^*_{NK_{Lx}} /A/_s$ and $\Gamma \vdash^*_{NK_{Lx}} \backslash\neg A\backslash_s$.

L10.2 If s is 0 or appears in Γ, and $\Gamma \nvdash^*_{NK_{Lx}} \backslash\neg P\backslash_s$, then $\Gamma \cup \{/P/_s\}$ is consistent.

Reasoning as in L7.2.

L10.3 There is an enumeration of all the subscripted formulas, $\mathcal{P}_1\ \mathcal{P}_2 \ldots$ with access relations $s.t$.

Proof by construction as usual.

MAX Γ is S-MAXIMAL iff for any A_s either $\Gamma \vdash^*_{NK_{Lx}} /A/_s$ or $\Gamma \vdash^*_{NK_{Lx}} \backslash\neg A\backslash_s$.

SGT Γ is a SCAPEGOAT set iff for every formula of the form $/\neg\Box A/_s$, if $\Gamma \vdash^*_{NK_{Lx}} /\neg\Box A/_s$ then there is some t such that $\Gamma \vdash^*_{NK_{Lx}} s.t$ and $\Gamma \vdash^*_{NK_{Lx}} /\neg A/_t$.

C(Γ') For Γ with unsubscripted formulas and the corresponding Γ_0, we construct Γ' as follows. Set $\Omega_0 = \Gamma_0$. By L10.3, there is an enumeration, $\mathcal{P}_1, \mathcal{P}_2 \ldots$ of all the formulas, together with all the access relations $s.t$; let $\mathcal{E}_0$ be this enumeration. Then for the first expression $\mathcal{P}$ in $\mathcal{E}_{i-1}$ such that all its subscripts are 0 or introduced in Ω_{i-1}, let $\mathcal{E}_i$ be like $\mathcal{E}_{i-1}$ but without $\mathcal{P}$, and set,

$$\begin{array}{ll} \Omega_i = \Omega_{i-1} & \text{if } \Omega_{i-1} \vdash^*_{NK_{Lx}} \backslash\neg A\backslash_s \\ \Omega_{i^*} = \Omega_{i-1} \cup \{/A/_s\} & \text{if } \Omega_{i-1} \nvdash^*_{NK_{Lx}} \backslash\neg A\backslash_s \end{array}$$

and

$$\begin{array}{ll} \Omega_i = \Omega_{i^*} & \text{if } /A/_s \text{ is not of the form } /\neg\Box P/_s \\ \Omega_i = \Omega_{i^*} \cup \{s.t, /\neg P/_t\} & \text{if } /A/_s \text{ is of the form } /\neg\Box P/_s \end{array}$$

-where t is the first subscript not included in Ω_{i^*}

then

$$\Gamma' = \bigcup_{i \geq 0} \Omega_i$$

Note that there is always sure to be a subscript t not in Ω_{i^*} insofar as there are infinitely many subscripts, and at any stage only finitely many formulas are added – the only subscripts in the initial Ω_0 being 0. Suppose s appears in Γ'; then there is some Ω_i in which it is first appears; and any formula $\mathcal{P}_j$ in the original enumeration that has subscript s is sure to be "considered" for inclusion at a subsequent stage.

L10.4 For any s included in Γ', Γ' is s-maximal.

Reasoning as in L7.4.

L10.5 If Γ_0 is consistent, then each Ω_i is consistent.

Suppose Γ_0 is consistent.

Basis: $\Omega_0 = \Gamma_0$ and Γ_0 is consistent; so Ω_0 is consistent.

Assp: For any $i, 0 \leq i < k$, Ω_i is consistent.

Show: Ω_k is consistent.

Ω_k is either (i) Ω_{k-1}, or (ii) $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$ or (iii) $\Omega_{k^*} \cup \{s.t, /\neg P/_t\}$.

(i) Suppose Ω_k is Ω_{k-1}. By assumption, Ω_{k-1} is consistent; so Ω_k is consistent.

(ii) Suppose Ω_k is $\Omega_{k^*} = \Omega_{k-1} \cup \{/A/_s\}$. Then by construction, s is 0 or in Ω_{k-1} and $\Omega_{k-1} \nvdash^*_{NK_{L\alpha}} \backslash\neg A\backslash_s$; so by L10.2, $\Omega_{k-1} \cup \{/A/_s\}$ is consistent; so Ω_k is consistent.

(iii) Suppose Ω_k is $\Omega_{k^*} \cup \{s.t, /\neg P/_t\}$. In this case, as above, Ω_{k*} is consistent and by construction, $/\neg\Box P/_s \in \Omega_{k^*}$. Suppose Ω_k is inconsistent. Then there are $/\!\!/A/\!\!/_u$ and $\backslash\!\!\backslash\neg A\backslash\!\!\backslash_u$ such that $\Omega_{k^*} \cup \{s.t, /\neg P/_t\} \vdash^*_{NK_{L\alpha}} /\!\!/A/\!\!/_u$ and $\Omega_{k^*} \cup \{s.t, /\neg P/_t\} \vdash^*_{NK_{L\alpha}} \backslash\!\!\backslash\neg A\backslash\!\!\backslash_u$. So reason as follows,

$$\begin{array}{r|lll}
1 & \Omega_{k^*} & & \\
2 & \mid\; \underline{s.t} & & \text{A } (g, \Box\text{I}) \\
3 & \mid\; \mid\; \underline{/\neg P/_t} & & \text{A } (c, \neg\text{E}) \\
4 & \mid\; \mid\; /\!\!/A/\!\!/_u & & \text{from } \Omega_{k^*} \cup \{s.t, /\neg P/_t\} \\
5 & \mid\; \mid\; \backslash\!\!\backslash\neg A\backslash\!\!\backslash_u & & \text{from } \Omega_{k^*} \cup \{s.t, /\neg P/_t\} \\
6 & \mid\; \backslash P\backslash_t & & \text{3-5 } \neg\text{E} \\
7 & \backslash\Box P\backslash_s & & \text{2-6 } \Box\text{I}
\end{array}$$

where, by construction, t is not in Ω_{k^*}. So $\Omega_{k^*} \vdash^*_{NK_{L\alpha}} \backslash\Box P\backslash_s$; but $/\neg\Box P/_s \in \Omega_{k^*}$; so $\Omega_{k^*} \vdash^*_{NK_{L\alpha}} /\neg\Box P/_s$; so Ω_{k^*} is inconsistent. This is impossible; reject the assumption: Ω_k is consistent.

For any i, Ω_i is consistent.

L10.6 If Γ_0 is consistent, then Γ' is consistent.

Reasoning parallel to L2.6 and L6.6.

L10.7 If Γ_0 is consistent, then Γ' is a scapegoat set.

Suppose Γ_0 is consistent and $\Gamma' \vdash^*_{NK_{L\alpha}} /\neg\Box P/_s$. By L10.6, Γ' is consistent; and by the constraints on subscripts, s is included in Γ'. Since Γ' is consistent, $\Gamma' \nvdash^*_{NK_{L\alpha}} \backslash\neg\neg\Box P\backslash_s$; so there is a stage in the construction process where $\Omega_{i^*} = \Omega_{i-1} \cup \{/\neg\Box P/_s\}$ and $\Omega_i = \Omega_{i^*} \cup \{s.t, /\neg P/_t\}$; so by construction, $s.t \in \Gamma'$ and $/\neg P/_t \in \Gamma'$; so $\Gamma' \vdash^*_{NK_{L\alpha}} s.t$ and $\Gamma' \vdash^*_{NK_{L\alpha}} /\neg P/_t$. So Γ' is a scapegoat set.

C(I) We construct an interpretation $I = \langle W, R, h\rangle$ based on Γ' as follows. Let W have a member w_s corresponding to each subscript s included in Γ'. Then set $\langle w_s, w_t\rangle \in R$ iff $\Gamma' \vdash^*_{NK_{L\alpha}} s.t$ and $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{NK_{L\alpha}} /p/_s$.

L10.8 If Γ_0 is consistent then for $\langle W, R, h\rangle$ constructed as above, and for any s included in Γ', $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NK_{L\alpha}} /A/_s$.

Suppose Γ_0 is consistent and s is included in Γ'. By L10.4, Γ' is s-maximal. By L10.6 and L10.7, Γ' is consistent and a scapegoat set. Now by induction on the number of operators in $/A/_s$,

Basis: If $/A/_s$ has no operators, then it is a parameter $/p/_s$ and by construction, $h_{w_s}(/p/) = 1$ iff $\Gamma' \vdash^*_{NK_{L\alpha}} /p/_s$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NK_{L\alpha}} /A/_s$.

Assp: For any i, $0 \leq i < k$, if $/A/_s$ has i operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NK_{L\alpha}} /A/_s$.

Show: If $/A/_s$ has k operators, then $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^*_{NK_{L\alpha}} /A/_s$.

If $/A/_s$ has k operators, then it is of the form $/\neg P/_s$, $/P \wedge Q/_s$, $/P \vee Q/_s$, $/\Box P/_s$, or $/\Diamond P/_s$, where P and Q have $< k$ operators.

($\neg$)

($\wedge$)

($\vee$)

($\Box$) $/A/_s$ is $/\Box P/_s$. (i) Suppose $h_{w_s}(/A/) = 1$ but $\Gamma' \nvdash^*_{NK_{L\alpha}} /A/_s$; then $h_{w_s}(/\Box P/) = 1$ but $\Gamma' \nvdash^*_{NK_{L\alpha}} /\Box P/_s$. From the latter, by s-maximality, $\Gamma' \vdash^*_{NK_{L\alpha}} \backslash\neg\Box P\backslash_s$; so, since Γ' is a scapegoat set, there is some t such that $\Gamma' \vdash^*_{NK_{L\alpha}} s.t$ and $\Gamma' \vdash^*_{NK_{L\alpha}} /\neg P/_t$; from the first, by construction, $\langle w_s, w_t\rangle \in R$; and from the second, by consistency, $\Gamma' \nvdash^{*}_{NK_{Lx}} /P/_t$; so by assumption, $h_{w_t}(/P/) = 0$; but w_sRw_t; so by $\mathrm{HK_L}(\Box)$, $h_{w_s}(/\Box P/) = 0$. This is impossible; reject the assumption: if $h_{w_s}(/A/) = 1$, then $\Gamma' \vdash^{*}_{NK_{Lx}} /A/_s$.

(ii) Suppose $\Gamma' \vdash^{*}_{NK_{Lx}} /A/_s$ but $h_{w_s}(/A/) = 0$; then $\Gamma' \vdash^{*}_{NK_{Lx}} /\Box P/_s$ but $h_{w_s}(/\Box P/) = 0$. From the latter, by $\mathrm{HK_L}(\Box)$, there is some $w_t \in W$ such that w_sRw_t and $h_{w_t}(/P/) = 0$; so by assumption, $\Gamma' \nvdash^{*}_{NK_{Lx}} /P/_t$; but since w_sRw_t, by construction, $\Gamma' \vdash^{*}_{NK_{Lx}} s.t$; so by ($\Box$E), $\Gamma' \vdash^{*}_{NK_{Lx}} /P/_t$. This is impossible; reject the assumption: if $\Gamma' \vdash_{NK_{Lx}} /A/_s$ then $h_{w_s}(/A/) = 1$. So $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^{*}_{NK_{Lx}} /A/_s$.

($\Diamond$)

For any A_s, $h_{w_s}(/A/) = 1$ iff $\Gamma' \vdash^{*}_{NK_{Lx}} /A/_s$.

L10.9 If Γ_0 is consistent, then $\langle W, R, h\rangle$ constructed as above is an K_{Lx} interpretation.

Reasoning parallel to L7.9*a*.

MAP For any $w_s \in W$, set $m(s) = w_s$; otherwise $m(s)$ is arbitrary.

L10.10 If Γ_0 is consistent, then $h_m(\Gamma_0) = 1$.

Reasoning parallel to L2.10 and L6.9.

Main result: Suppose $\Gamma \models_{K_{Lx}} A$ but $\Gamma \nvdash_{NK_{Lx}} A$. Then $\Gamma_0 \models^{*}_{K_{Lx}} A_0$ but $\Gamma_0 \nvdash^{*}_{NK_{Lx}} A_0$. By (DN), if $\Gamma_0 \vdash^{*}_{NK_{Lx}} \neg\neg A_0$, then $\Gamma_0 \vdash^{*}_{NK_{Lx}} A_0$; so $\Gamma_0 \nvdash_{NK_{Lx}}$ $*]\neg\neg A_0$; so by L10.2, $\Gamma_0 \cup \{\overline{\neg A_0}\}$ is consistent; so by L10.9 and L10.10, there is an K_{Lx} interpretation $\langle W, R, h\rangle_m$ constructed as above such that $h_m(\Gamma_0 \cup \{\overline{\neg A_0}\}) = 1$; so $h_{m(0)}(\overline{\neg A}) = 1$; so by $\mathrm{HK_L}(\neg)$, $h_{m(0)}(A) = 0$; so $h_m(\Gamma_0) = 1$ and $h_{m(0)}(A) = 0$; so by $\mathrm{VK_L}*$, $\Gamma_0 \not\models^{*}_{K_{Lx}} A_0$. This is impossible; reject the assumption: if $\Gamma \models_{K_{Lx}} A$, then $\Gamma \vdash_{NK_{Lx}} A$.

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