Notes Toward Completion of the American Plan

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This document proposes a "completion" of the American Plan alternative to that in (3). It is striking that the difficulty for the four-valued approach occurs occurs at precisely the same location as that addressed for the simplified semantics in (2). The current proposal is particularly natural against the background of solution there proposed.

I. LANGUAGE / BASIC SEMANTIC NOTIONS

L4 The *vocabulary* consists of propositional parameters $p_0, p_1 \ldots$ with the operators $\neg$, $\wedge$, and $\rightarrow$. Each propositional parameter is a *formula*; if A and B are formulas, so are $\neg A$, $(A \wedge B)$ and $(A \rightarrow B)$. Other operators abbreviate in the usual way. If A is a formula so formed, so is $\overline{A}$.

Let $/A/$ and $\backslash A\backslash$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $/A/$ and $\backslash A\backslash$ are opposite. And similarly for other expressions with overlines as below.

I4 An *interpretation* for the basic logic DW is $\langle W, N, \overline{N}, R, \overline{R}, h\rangle$ where W is a set of worlds; $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively; $R, \overline{R} \subseteq W^3$ are access relations for truth and non-falsity respectively; and h is a function which assigns 1 or 0 to each $/p/$ at each $w \in W$. When $h_w(/p/) = 1$ we say $/p/$ *holds* at w and otherwise *fails*. As a constraint on interpretations we require also,

NC For any $w \in /N/$, $w/R/xy$ iff $x = y$

Where x is empty or includes additional constraints as described below, a *4x* interpretation incorporates also any constraints in x.

H4 For complex expressions,

(¬) $h_w(/\neg P/) = 1$ iff $h_w(\backslash P\backslash) = 0$

(∧) $h_w(/P \wedge Q/) = 1$ iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$

(→) $h_w(/P \rightarrow Q/) = 1$ iff there are no $x, y \in W$ such that $w/R/xy$ and either $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$

For a set Γ of formulas, $h_w(\Gamma) = 1$ iff $h_w(/P/) = 1$ for each $/P/ \in \Gamma$; then,

V4 $\Gamma \models_{4x} P$ iff there is no 4x interpretation $\langle W, N, \overline{N}, R, \overline{R}, h\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

Other logics may be obtained by placing constraints on access. Numbering of conditions is kept mostly parallel to (1).

BASIC CONSTRAINTS

D3/4 If $a/R/bx$ and $xRcd$ then there is a y such that $bRcy$ and $a/R/yd$, and a z such that $bRzd$ and $a/R/cz$. And if $a/R/xb$ and $x\overline{R}cd$ then there is a y such that $b\overline{R}cy$ and $a/R/yd$, and a z such that $b\overline{R}zd$ and $a/R/cz$.

D5 If $a/R/bc$ then there is a y such that $a/R/by$ and $yRbc$ and a z such that $a/R/zc$ and $z\overline{R}bc$.

We have the effect of D20 'for free' from the double condition from H4(→).

PERMUTATION: As explained in (2) the standard simplified semantics requires a notion of *inclusion* in order to accommodate permutation. Insofar as we are working with a parallel version of the four-valued approach, something similar is required here. In this case, a family of inclusion notions is available and required.

$$a \leq b \Rightarrow \begin{cases} \text{if } h_a(p) = 1 \text{ then } h_b(p) = 1 \text{ and if } h_b(\overline{p}) = 1 \text{ then } h_a(\overline{p}) = 1 \\ \text{if } bRxy \text{ then } aRxy \text{ if } a \notin N\text{, otherwise if } bRxy \text{ then } x \leq y \\ \text{if } a\overline{R}xy \text{ then } b\overline{R}xy \text{ if } b \notin \overline{N}\text{, otherwise if } a\overline{R}xy \text{ then } x \leq y \end{cases}$$

$$a \leq^* b \Rightarrow \begin{cases} \text{if } h_a(p) = 1 \text{ then } h_b(\overline{p}) = 1 \text{ and if } h_b(p) = 1 \text{ then } h_a(\overline{p}) = 1 \\ \text{if } b\overline{R}xy \text{ then } aRxy \text{ if } a \notin N\text{, otherwise if } b\overline{R}xy \text{ then } x \leq y \\ \text{if } a\overline{R}xy \text{ then } bRxy \text{ if } b \notin N\text{, otherwise if } a\overline{R}xy \text{ then } x \leq y \end{cases}$$

$$a \leq^\sharp b \Rightarrow \begin{cases} \text{if } h_a(\overline{p}) = 1 \text{ then } h_b(p) = 1 \text{ and if } h_b(\overline{p}) = 1 \text{ then } h_a(p) = 1 \\ \text{if } bRxy \text{ then } a\overline{R}xy \text{ if } a \notin \overline{N}\text{, otherwise if } bRxy \text{ then } x \leq y \\ \text{if } aRxy \text{ then } b\overline{R}xy \text{ if } b \notin \overline{N}\text{, otherwise if } aRxy \text{ then } x \leq y \end{cases}$$

Given this we have,

D6 If $aRbc$ then for some $y \geq a$, $bRyc$, and for some $z \geq^* a$, $c\overline{R}bz$. And if $a\overline{R}bc$ then for some $y \geq^\sharp a$, $bRyc$, and for some $z \leq a$, $c\overline{R}bz$

DW takes none of the extra constraints. TW has D3/4; RW D3/4 and D6; and R has D3/4, D5 and D6. Thus the American Plan is 'completed' at least through R.

II. SOUNDNESS

BASIC PRINCIPLES: We show that the semantics is adequate for basic axioms and rules by direct arguments, a few of which are worked. Arguments are typically case heavy but quite parallel, and collapsed to some extent by the notation.

A1. $A \to A$

A2. $A \to (A \vee B)$ and $B \to (A \vee B)$

A3. $(A \wedge B) \to A$ and $(A \wedge B) \to B$

A4. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee C]$

A5. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A6. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A7. $\neg\neg A \to A$

R1. $A, A \to B$ so B

R2. A, B so $A \wedge B$

R3. $A \to B, C \to D$ so $(B \to C) \to (A \to D)$

C20. $(A \to \neg B) \to (B \to \neg A)$

Suppose $\not\models_4 (A \to \neg B) \to (B \to \neg A)$; then there is a $\langle W, N, \overline{N}, R, \overline{R}, h\rangle$ and $w \in N$ such that $h_w(A \to \neg B) \to (B \to \neg A)) = 0$. Since $w \in N$, with NC there is some $a \in W$ s.t. $h_a(/A \to \neg B/) = 1$ and $h_a(/B \to \neg A/) = 0$. From the latter, there are $x, y \in W$ s.t. $a/R/xy$ and either $h_x(B) = 1$ and $h_y(\neg A) = 0$, or $h_y(\overline{B}) = 1$ and $h_x(\overline{\neg A}) = 0$. In the first case, $h_y(\neg A) = 0$; so $h_y(\overline{A}) = 1$; so with $a/R/xy$ and $h_a(/A \to \neg B/) = 1$, $h_x(\overline{\neg B}) = 1$; so $h_x(B) = 0$; which contradicts $h_x(B) = 1$. In the second case, $h_x(\overline{\neg A}) = 0$; so $h_x(A) = 1$; so with $a/R/xy$ and $h_a(/A \to \neg B/) = 1$, $h_y(\neg B) = 1$; so $h_y(\overline{B}) = 0$; which contradicts $h_y(\overline{B}) = 1$.

C3. $(A \to B) \to [(B \to C) \to (A \to C)]$ – given D3/4

Suppose $\not\models_{4x} (A \to B) \to [(B \to C) \to (A \to C)]$; then there is a $\langle W, N, \overline{N}, R, \overline{R}, h\rangle$ and $w \in N$ such that $h_w((A \to B) \to [(B \to C) \to (A \to C)]) = 0$. Since $w \in N$, with NC there is some $a \in W$ s.t. $h_a(/A \to B/) = 1$ and $h_a(/(B \to C) \to (A \to C)/) = 0$. From the latter, there are $b, c \in W$ s.t. $a/R/bc$ and either (i) or (ii). (i) $h_b(B \to C) = 1$ and $h_c(A \to C) = 0$. From the latter, there are $d, e \in W$ s.t. $cRde$ and either $h_d(A) = 1$ and $h_e(C) = 0$, or $h_e(\overline{A}) = 1$ and $h_d(\overline{C}) = 0$. In the first case, from $a/R/bc$ and $cRde$ by D3/4 there is a z s.t. $bRze$ and $a/R/dz$; so with $h_a(/A \to B/) = 1$ and $h_d(A) = 1$, $h_z(B) = 1$; so with $h_b(B \to C) = 1$, $h_e(C) = 1$ which contradicts $h_e(C) = 0$. In the second case, from $a/R/bc$ and $cRde$ by D3/4 there is a y s.t. $bRdy$ and $a/R/ye$; so with $h_a(/A \to B/) = 1$ and $h_e(\overline{A}) = 1$, $h_y(\overline{B}) = 1$; so with $h_b(B \to C) = 1$, $h_d(\overline{C}) = 1$; which contradicts $h_d(\overline{C}) = 0$. (ii) $h_c(\overline{B \to C}) = 1$ and $h_b(\overline{A \to C}) = 0$. From the latter, there are $d, e \in W$ s.t. $b\overline{R}de$ and either $h_d(A) = 1$ and $h_e(C) = 0$, or $h_e(\overline{A}) = 1$ and $h_d(\overline{C}) = 0$. In the first case, from $a/R/bc$ and $b\overline{R}de$ by D3/4 there is a z s.t. $c\overline{R}ze$ and $a/R/dz$; so with $h_a(/A \to B/) = 1$ and $h_d(A) = 1$, $h_z(B) = 1$; so with $h_c(\overline{B \to C}) = 1$, $h_e(C) = 1$

which contradicts $h_e(C) = 0$. In the second case, from $a/R/bc$ and $b\overline{R}de$ by D3/4 there is a y s.t. $c\overline{R}dy$ and $a/R/ye$; so with $h_a(/A \to B/) = 1$ and $h_e(\overline{A}) = 1$, $h_y(\overline{B}) = 1$; so with $h_c(\overline{B \to C}) = 1$, $h_d(\overline{C}) = 1$; which contradicts $h_d(\overline{C}) = 0$.

C4. $(A \to B) \to [(C \to A) \to (C \to B)]$ – given D3/4

C5. $[A \to (A \to B)] \to (A \to B)$ – given D5

Suppose $\not\models_{4x} [A \to (A \to B)] \to (A \to B)$; then there is a $\langle W, N, \overline{N}, R, \overline{R}, h\rangle$ and $w \in N$ such that $h_w([A \to (A \to B)] \to (A \to B)) = 0$. Since $w \in N$, with NC there is some $a \in W$ s.t. $h_a(/A \to (A \to B)/) = 1$ and $h_a(/A \to B/) = 0$. From the latter, there are $b, c \in W$ s.t. $a/R/bc$ and either $h_b(A) = 1$ and $h_c(B) = 0$, or $h_c(\overline{A}) = 1$ and $h_b(\overline{B}) = 0$. Consider the first case. From $a/R/bc$ with D5, there is a y such that $a/R/by$ and $yRbc$; so with $h_a(/A \to (A \to B)/) = 1$ and $h_b(A) = 1$, $h_y(A \to B) = 1$; so $h_c(B) = 1$; which contradicts $h_c(B) = 0$. Consider the second case. From $a/R/bc$ with D5, there is a z such that $a/R/zc$ and $z\overline{R}bc$; so with with $h_a(/A \to (A \to B)/) = 1$ and $h_c(\overline{A}) = 1$, $h_z(\overline{A \to B}) = 1$; so $h_b(\overline{B}) = 1$; which contradicts $h_b(\overline{B}) = 0$.

PERMUTATION: For this we need that the inclusion relations preserve generally conditions on atomics. As the arguments are similar, I work just the first.

L1 For any $a, b \in W$, if $a \leq b$, then (i) if $h_a(P) = 1$, then $h_b(P) = 1$ and (ii) if $h_b(\overline{P}) = 1$ then $h_a(\overline{P}) = 1$.

Suppose $a \leq b$; then it is immediate that an atomic is such that (i) and (ii). So suppose that if $a \leq b$ then A and B satisfy (i) and (ii).

Suppose P is $\neg A$ and $a \leq b$. (i) Suppose $h_a(\neg A) = 1$; then $h_a(\overline{A}) = 0$; so by assp. $h_b(\overline{A}) = 0$; so $h_b(\neg A) = 1$. (ii) Suppose $h_b(\overline{\neg A}) = 1$; then $h_b(A) = 0$; so by assp. $h_a(A) = 0$; so $h_a(\overline{\neg A}) = 1$. And similarly for $(\wedge)$.

Suppose P is $A \to B$ and $a \leq b$. (i) Suppose $h_a(A \to B) = 1$; we want to show $h_b(A \to B) = 1$. Suppose otherwise, that $h_b(A \to B) = 0$; then there are $x, y \in W$ s.t. $bRxy$ and either (1) $h_x(A) = 1$ and $h_y(B) = 0$, or (2) $h_y(\overline{A}) = 1$ and $h_x(\overline{B}) = 0$. We consider these in two cases: (a) $a \notin N$; then from $bRxy$ and $a \leq b$ we have $aRxy$. (1) $h_x(A) = 1$; so with $h_a(A \to B) = 1$, we have $h_y(B) = 1$, which contradicts $h_y(B) = 0$. (2) $h_y(\overline{A}) = 1$; so with $h_a(A \to B) = 1$, we have $h_x(\overline{B}) = 1$, which contradicts $h_x(\overline{B}) = 0$. (b) $a \in N$; then from $bRxy$ and $a \leq b$ we have $x \leq y$. (1) $h_x(A) = 1$; so with $a \in N$ and $h_a(A \to B) = 1$, $h_a(B) = 1$; so with $x \leq y$ by assp. $h_y(B) = 1$, which contradicts $h_y(B) = 0$. (2) $h_y(\overline{A}) = 1$; so with $a \in N$ and $h_a(A \to B) = 1$, $h_y(\overline{B}) = 1$; so with $x \leq y$ by assp. $h_x(\overline{B}) = 1$, which contradicts $h_x(\overline{B}) = 0$. And similarly for (ii).

L2 For any $a, b \in W$, if $a \leq^* b$, then (i) if $h_a(P) = 1$, then $h_b(\overline{P}) = 1$ and (ii) if $h_b(P) = 1$ then $h_a(\overline{P}) = 1$.

L3 For any $a, b \in W$, if $a \leq^\sharp b$, then (i) if $h_a(\overline{P}) = 1$, then $h_b(P) = 1$ and (ii) if $h_b(\overline{P}) = 1$ then $h_a(P) = 1$.

Now we are in a position to argue for C6 in the usual way.

C6. $A \to [(A \to B) \to B]$ – given D6

Suppose $\not\models_{4v} A \to [(A \to B) \to B]$; then there is a $\langle W, N, \overline{N}, R, \overline{R}, h\rangle$ and $w \in N$ s.t. $h_w(A \to [(A \to B) \to B]) = 0$. Since $w \in N$, with NC there is some $a \in W$ s.t. either (i) or (ii). (i) $h_a(A) = 1$ and $h_a((A \to B) \to B) = 0$. From the latter, there are $b, c \in W$ such that $aRbc$ and either $h_b(A \to B) = 1$ and $h_c(B) = 0$, or $h_c(\overline{A \to B}) = 1$ and $h_b(\overline{B}) = 0$. Suppose the first of these; from $aRbc$ by D6 there is a $y \geq a$ s.t. $bRyc$; from $h_a(A) = 1$ and $y \geq a$, by L1, $h_y(A) = 1$; so with $bRyc$ and $h_b(A \to B) = 1$, $h_c(B) = 1$, which contradicts $h_c(B) = 0$. Suppose the second, from $aRbc$ by D6 there is a $z \geq^* a$ s.t. $c\overline{R}bz$; from $h_a(A) = 1$ and $z \geq^* a$, by L2, $h_z(\overline{A}) = 1$; so with $c\overline{R}bz$ and $h_c(\overline{A \to B}) = 1$, $h_b(\overline{B}) = 1$, which contradicts $h_b(\overline{B}) = 0$. (ii) $h_a(\overline{A}) = 1$ and $h_a(\overline{(A \to B) \to B}) = 0$. From the latter, there are $b, c \in W$ such that $a\overline{R}bc$ and either $h_b(A \to B) = 1$ and $h_c(B) = 0$, or $h_c(\overline{A \to B}) = 1$ and $h_b(\overline{B}) = 0$. Suppose the first; from $a\overline{R}bc$ by D6 there is a $y \geq^\sharp a$ s.t. $bRyc$; from $h_a(\overline{A}) = 1$ and $y \geq^\sharp a$, by L3, $h_y(A) = 1$; so with $bRyc$ and $h_b(A \to B) = 1$, $h_c(B) = 1$, which contradicts $h_c(B) = 0$. Suppose the second, from $a\overline{R}bc$ by D6 there is a $z \leq a$ s.t. $c\overline{R}bz$; from $h_a(\overline{A}) = 1$ and $z \leq a$, by L1, $h_z(\overline{A}) = 1$; so with $c\overline{R}bz$ and $h_c(\overline{A \to B}) = 1$, $h_b(\overline{B}) = 1$, which contradicts $h_b(\overline{B}) = 0$.

III. COMPLETENESS

Rather than show completeness directly, we set out to show that for any interpretation on the simplified semantics is a corresponding four-valued interpretation that preserves all the same truths. Completeness then follows directly from completeness on the simplified semantics.

BASIC SIMPLIFIED SEMANTICS: An *interpretation* is $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \mathsf{v}\rangle$ where W is a set of worlds; $\mathsf{g} \in \mathsf{W}$; $\mathsf{R} \subseteq \mathsf{W}^3$; $\star$ a function from W to W; and v a function such that for any $\mathsf{w} \in \mathsf{W}$ and p, $\mathsf{v_w}(p) = 1$ or $\mathsf{v_w}(p) = 0$. Let $\leq$ be a reflexive, transitive relation on W such that if $\mathsf{a} \leq \mathsf{b}$ then $\mathsf{a} \trianglelefteq \mathsf{b}$ and $\mathsf{b}^\star \trianglelefteq \mathsf{a}^\star$, where,[1]

$$a \trianglelefteq b = \begin{cases} \text{if } \mathsf{v_a}(p) = 1 \text{ then } \mathsf{v_b}(p) = 1 \\ \text{if } \mathsf{bRxy} \text{ and } \mathsf{a} \neq \mathsf{g}, \text{ then } \mathsf{aRxy} \\ \text{if } \mathsf{bRxy} \text{ and } \mathsf{a} = \mathsf{g} \text{ then } \mathsf{x} \leq \mathsf{y} \end{cases}$$

[1] Observe that this diverges from the definition in (1) and (2) where $\mathsf{a} \leq \mathsf{b}$ requires $\mathsf{b}^\star \leq \mathsf{a}^\star$, which in turn requires $\mathsf{a}^{\star\star} \leq \mathsf{b}^{\star\star}$, etc. Given restrictions on $\star$, the accounts seem effectively the same, though the above above simplifies contact with other inclusion relations.

Then as constraints on interpretations, we require also, S: for any $\mathsf{a} \in \mathsf{W}$, $\mathsf{a} = \mathsf{a}^{\star\star}$; NC: for any $\mathsf{a}, \mathsf{b} \in \mathsf{W}$, gRab iff $\mathsf{a} = \mathsf{b}$; and $\mathsf{D20}$: for any $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathsf{W}$, if aRbc then $\mathsf{aRc^{\star}b^{\star}}$. Where x is empty or includes additional constraints as described below, an Sx interpretation incorporates also any constraints in x.

(¬) $\mathsf{v_w}(\neg A) = 1$ iff $\mathsf{v_{w^{\star}}}(A) = 0$

(∧) $\mathsf{v_w}(A \wedge B) = 1$ iff $\mathsf{v_w}(A) = 1$ and $\mathsf{v_w}(B) = 1$

(→) $\mathsf{v_w}(A \rightarrow B) = 1$ iff there are no $\mathsf{x}, \mathsf{y} \in \mathsf{W}$ such that wRxy and $\mathsf{v_x}(A) = 1$ but $\mathsf{v_y}(B) = 0$

For a set Γ of formulas, $\mathsf{v_w}(\Gamma) = 1$ iff $\mathsf{v_w}(A) = 1$ for each $A \in \Gamma$; then,

VSx $\Gamma \models_{Sx} A$ iff there is no Sx interpretation $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \mathsf{v} \rangle$ such that $\mathsf{v_g}(\Gamma) = 1$ and $\mathsf{v_g}(A) = 0$.

OPTIONAL CONSTRAINTS: We require matched clauses for $\mathsf{D5}$ and $\mathsf{D6}$. In each case, these follow immediately the presence of $\mathsf{D20}$ and S.

D3/4 If there is an x such that aRbx and xRcd then there is a y such that aRcy and bRyd and there is a z such that bRcz and aRzd

D5 If aRbc then there is a y such that aRby and yRbc (and there is a $\mathsf{z^{\star}}$ such that $\mathsf{aRc^{\star}z^{\star}}$ and $\mathsf{z^{\star}Rc^{\star}b^{\star}}$)

Suppose aRbc; then by $\mathsf{D20}$, $\mathsf{aRc^{\star}b^{\star}}$; so there is a y such that $\mathsf{aRc^{\star}y}$ and $\mathsf{yRc^{\star}b^{\star}}$; set $\mathsf{z} = \mathsf{y^{\star}}$; then $\mathsf{z^{\star}} = \mathsf{y^{\star\star}} = \mathsf{y}$; so if aRbc there is a $\mathsf{z^{\star}}$ such that $\mathsf{aRc^{\star}z^{\star}}$ and $\mathsf{z^{\star}Rc^{\star}b^{\star}}$.

D6 If aRbc then there is a $\mathsf{y} \geq \mathsf{a}$ such that bRyc, (and there is a $\mathsf{z^{\star}} \geq \mathsf{a}$ such that $\mathsf{c^{\star}Rz^{\star}b^{\star}}$).

Suppose aRbc; then by $\mathsf{D20}$, $\mathsf{aRc^{\star}b^{\star}}$; so there is a $\mathsf{y} \geq \mathsf{a}$ such that $\mathsf{c^{\star}Ryb^{\star}}$; set $\mathsf{z} = \mathsf{y^{\star}}$; then $\mathsf{z^{\star}} = \mathsf{y^{\star\star}} = \mathsf{y}$; so if aRbc, there is a $\mathsf{z^{\star}} \geq \mathsf{a}$ such that $\mathsf{c^{\star}Rz^{\star}b^{\star}}$.

CONSTRUCTION AND RESULTS: For any $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \mathsf{v} \rangle$ consider a corresponding $\langle W, N, \overline{N}, R, \overline{R}, h \rangle$ such that there is a $w \in W$ corresponding to each $\mathsf{w} \in \mathsf{W}$, where $N = \{g\}$; $\overline{N} = \{w \mid \mathsf{w^{\star}} = \mathsf{g}\}$; $R = \{\langle x, y, z \rangle \mid \langle \mathsf{x}, \mathsf{y}, \mathsf{z} \rangle \in \mathsf{R}\}$; $\overline{R} = \{\langle x, y, z \rangle \mid \langle \mathsf{x^{\star}}, \mathsf{y}, \mathsf{z} \rangle \in \mathsf{R}\}$; $h_w(p) = \mathsf{v_w}(p)$; and $h_w(\overline{p}) = \mathsf{v_{w^{\star}}}(p)$. And set $a \leq b$ iff $\mathsf{a} \trianglelefteq \mathsf{b}$ and $\mathsf{b^{\star}} \trianglelefteq \mathsf{a^{\star}}$; $a \leq^{*} b$ iff $\mathsf{a} \trianglelefteq \mathsf{b^{\star}}$ and $\mathsf{b} \trianglelefteq \mathsf{a^{\star}}$; and $a \leq^{\sharp} b$ iff $\mathsf{a^{\star}} \trianglelefteq \mathsf{b}$ and $\mathsf{b^{\star}} \trianglelefteq \mathsf{a}$.

L4 If $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \mathsf{v} \rangle$ is an Sx interpretation then $\langle W, N, \overline{N}, R, \overline{R}, h \rangle$ constructed as above is a 4x interpretation such that if $\mathsf{Dn} \in \mathsf{x}$ then $Dn \in x$.

Suppose $w \in N$; then $w = g$. Say $wRxy$; then $gRxy$ and by construction, gRxy; so by NC, $\mathsf{x} = \mathsf{y}$; so $x = y$. Say $x = y$; then $\mathsf{x} = \mathsf{y}$; so by NC, gRxy; so by construction, $gRxy$, which is to say $wRxy$. Suppose $w \in \overline{N}$; then $\mathsf{w^{\star}} = \mathsf{g}$. Say $w\overline{R}xy$; then by construction, $\mathsf{w}^\star\mathsf{Rxy}$; so gRxy; so by NC, $\mathsf{x} = \mathsf{y}$; so $x = y$. Say $x = y$; then $\mathsf{x} = \mathsf{y}$; so by NC, gRxy; so $\mathsf{w}^\star\mathsf{Rxy}$; so by construction, $w\overline{R}xy$. So NC is satisfied.

Suppose $a \leq^\sharp b$; then $\mathsf{a}^\star \trianglelefteq \mathsf{b}$ and $\mathsf{b}^\star \trianglelefteq \mathsf{a}$. Say $h_a(\overline{p}) = 1$; then $\mathsf{v}_{\mathsf{a}^\star}(\mathsf{p}) = 1$ so by $\mathsf{a}^\star \trianglelefteq \mathsf{b}$, $\mathsf{v}_\mathsf{b}(\mathsf{p}) = 1$; so $h_b(p) = 1$. Suppose $h_b(\overline{p}) = 1$; then $\mathsf{v}_{\mathsf{b}^\star}(\mathsf{p}) = 1$; so by $\mathsf{b}^\star \trianglelefteq \mathsf{a}$, $\mathsf{v}_\mathsf{a}(\mathsf{p}) = 1$; so $h_a(p) = 1$. Suppose $bRxy$ and $a \notin \overline{N}$; then bRxy and $\mathsf{a}^\star \neq \mathsf{g}$; so by $\mathsf{a}^\star \trianglelefteq \mathsf{b}$, $\mathsf{a}^\star\mathsf{Rxy}$; so $a\overline{R}xy$. Suppose $bRxy$ and $a \in \overline{N}$; then bRxy and $\mathsf{a}^\star = \mathsf{g}$; so by $\mathsf{a}^\star \trianglelefteq \mathsf{b}$, $\mathsf{x} \leq \mathsf{y}$; so $x \leq y$. Suppose $aRxy$ and $b \notin \overline{N}$; then aRxy and $\mathsf{b}^\star \neq \mathsf{g}$; so by $\mathsf{b}^\star \trianglelefteq \mathsf{a}$, $\mathsf{b}^\star\mathsf{Rxy}$; so $b\overline{R}xy$. Suppose $aRxy$ and $b \in \overline{N}$; then aRxy and $\mathsf{b}^\star = \mathsf{g}$; so by $\mathsf{b}^\star \trianglelefteq \mathsf{a}$, $\mathsf{x} \leq \mathsf{y}$; so $x \leq y$. So $a \leq^\sharp b$ has the right form; and similarly for $a \leq b$ and $a \leq^* b$.

Suppose D3/4. Suppose $aRbx$ and $xRcd$. Then by construction, aRbx and xRcd; so by D3/4, there is a y such that aRcy and bRyd and there is a z such that bRcz and aRzd. So by construction, there is a y such that $aRcy$ and $bRyd$ and there is a z such that $bRcz$ and $aRzd$. Suppose $a\overline{R}bx$ and $xRcd$. Then by construction, $\mathsf{a}^\star\mathsf{Rbx}$ and xRcd; so by D3/4, there is a y such that $\mathsf{a}^\star\mathsf{Rcy}$ and bRyd and there is a z such that bRcz and $\mathsf{a}^\star\mathsf{Rzd}$. So by construction, there is a y such that $a\overline{R}cy$ and $bRyd$ and there is a z such that $bRcz$ and $a\overline{R}zd$. These satisfy D3/4. And similarly in other cases.

Suppose D5. Suppose $aRbc$; then by construction, aRbc; so by D5, there is a y such that aRby and yRbc, and there is a $\mathsf{z}^\star$ such that $\mathsf{aRc}^\star\mathsf{z}^\star$ and $\mathsf{z}^\star\mathsf{Rc}^\star\mathsf{b}^\star$; and with D20, aRzc and $\mathsf{z}^\star\mathsf{Rbc}$; so by construction, there is a y such that $aRby$ and $yRbc$, and there is a z such that $aRzc$ and $z\overline{R}bc$. Suppose $a\overline{R}bc$; then by construction, $\mathsf{a}^\star\mathsf{Rbc}$; so by D5, there is a y such that $\mathsf{a}^\star\mathsf{Rby}$ and yRbc, and there is a $\mathsf{z}^\star$ such that $\mathsf{a}^\star\mathsf{Rc}^\star\mathsf{z}^\star$ and $\mathsf{z}^\star\mathsf{Rc}^\star\mathsf{b}^\star$; and with D20, $\mathsf{a}^\star\mathsf{Rzc}$ and $\mathsf{z}^\star\mathsf{Rbc}$; so by construction, there is a y such that $a\overline{R}by$ and $yRbc$, and there is a z such that $a\overline{R}zc$ and $z\overline{R}bc$. In either case, D5 is satisfied.

Suppose D6. Suppose $aRbc$; then aRbc. By D6 there is a $\mathsf{y} \geq \mathsf{a}$ s.t. bRyc; so $bRyc$; and since $\mathsf{y} \geq \mathsf{a}$, $\mathsf{y} \trianglerighteq \mathsf{a}$ and $\mathsf{a}^\star \trianglerighteq \mathsf{y}^\star$; so $y \geq a$. And by D6 again there is a $\mathsf{z}^\star \geq \mathsf{a}$ s.t. $\mathsf{c}^\star\mathsf{Rz}^\star\mathsf{b}^\star$; and by D20, $\mathsf{c}^\star\mathsf{Rbz}$; so $c\overline{R}bz$; and since $\mathsf{z}^\star \geq \mathsf{a}$, $\mathsf{z}^\star \trianglerighteq \mathsf{a}$ and $\mathsf{a}^\star \trianglerighteq \mathsf{z}^{\star\star} = \mathsf{z}$; so $z \geq^* a$. Suppose $a\overline{R}bc$; then $\mathsf{a}^\star\mathsf{Rbc}$. By D6 there is a $\mathsf{y} \geq \mathsf{a}^\star$ s.t. bRyc; so $bRyc$; and since $\mathsf{y} \geq \mathsf{a}^\star$, $\mathsf{y} \trianglerighteq \mathsf{a}^\star$ and $\mathsf{a} = \mathsf{a}^{\star\star} \trianglerighteq \mathsf{y}^\star$; so $y \geq^\sharp a$. By D6 again there is a $\mathsf{z}^\star \geq \mathsf{a}^\star$ s.t. $\mathsf{c}^\star\mathsf{Rz}^\star\mathsf{b}^\star$; and by D20, $\mathsf{c}^\star\mathsf{Rbz}$; so $c\overline{R}bz$; and since $\mathsf{z}^\star \geq \mathsf{a}^\star$, $\mathsf{z}^\star \trianglerighteq \mathsf{a}^\star$ and $\mathsf{a} = \mathsf{a}^{\star\star} \trianglerighteq \mathsf{z}^{\star\star} = \mathsf{z}$; so $z \leq a$. So D6 is satisfied.

L5 Where $\langle \mathsf{W}, \mathsf{N}, \mathsf{R}, \star, \mathsf{v}\rangle$ and $\langle W, N, \overline{N}, R, \overline{R}, v\rangle$ are as above, for any A, (i) $h_w(A) = \mathsf{v}_\mathsf{w}(A)$ and (ii) $h_w(\overline{A}) = \mathsf{v}_{\mathsf{w}^\star}(A)$.

Suppose $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \mathsf{v}\rangle$ and $\langle W, N, \overline{N}, R, \overline{R}, v\rangle$ are as above. By construction, atomics are such that (i) and (ii). So suppose, P and Q are such that (i) and (ii).

Suppose A is $\neg P$. (i) $h_w(\neg P) = 1$ iff $h_w(\overline{P}) = 0$; by assumption, iff $\mathsf{v}_{\mathsf{w}^*}(P) = 0$; iff $\mathsf{v}_{\mathsf{w}}(\neg P) = 1$. (ii) $h_w(\overline{\neg P}) = 1$ iff $h_w(P) = 0$; by assumption, iff $\mathsf{v}_{\mathsf{w}}(P) = 0$; iff $\mathsf{v}_{\mathsf{w}^*}(\neg P) = 1$. And similarly for $\wedge$.

Suppose A is $P \to Q$. (i) Suppose $\mathsf{v}_{\mathsf{w}}(P \to Q) = 0$; then there are $\mathsf{x}, \mathsf{y} \in \mathsf{W}$ such that wRxy and $\mathsf{v}_{\mathsf{x}}(P) = 1$ but $\mathsf{v}_{\mathsf{y}}(Q) = 0$; so by assumption, $h_x(P) = 1$ but $h_y(Q) = 0$. And since wRxy, $wRxy$. So $v_w(P \to Q) = 0$. Suppose $v_w(P \to Q) = 0$; then there are $x, y \in W$ such that $wRxy$ and either $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$; so by assumption, either $\mathsf{v}_{\mathsf{x}}(P) = 1$ but $\mathsf{v}_{\mathsf{y}}(Q) = 0$, or $\mathsf{v}_{\mathsf{y}^\star}(P) = 1$ but $\mathsf{v}_{\mathsf{x}^\star}(Q) = 0$. And from $wRxy$, wRxy; so in the first case, $\mathsf{v}_{\mathsf{w}}(P \to Q) = 0$. But for the second case, we have by $\mathsf{D20}$, that $\mathsf{wRy^\star x^\star}$, and so that $\mathsf{v}_{\mathsf{w}}(P \to Q) = 0$.

(ii) Suppose $\mathsf{v}_{\mathsf{w}^\star}(P \to Q) = 0$; then there are $\mathsf{x}, \mathsf{y} \in \mathsf{W}$ such that $\mathsf{w^\star Rxy}$ and $\mathsf{v}_{\mathsf{x}}(P) = 1$ but $\mathsf{v}_{\mathsf{y}}(Q) = 0$; so by assumption, $h_x(P) = 1$ but $h_y(Q) = 0$. And since $\mathsf{w^\star Rxy}$, $w\overline{R}xy$. So $v_w(\overline{P \to Q}) = 0$. Suppose $v_w(\overline{P \to Q}) = 0$; then there are $x, y \in W$ such that $w\overline{R}xy$ and either $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$; so by assumption, either $\mathsf{v}_{\mathsf{x}}(P) = 1$ but $\mathsf{v}_{\mathsf{y}}(Q) = 0$, or $\mathsf{v}_{\mathsf{y}^\star}(P) = 1$ but $\mathsf{v}_{\mathsf{x}^\star}(Q) = 0$. And since $w\overline{R}xy$, $\mathsf{w^\star Rxy}$; so in the first case, $\mathsf{v}_{\mathsf{w}^\star}(P \to Q) = 0$. And for the second case, we have by $\mathsf{D20}$, that $\mathsf{w^\star Ry^\star x^\star}$, and so that $\mathsf{v}_{\mathsf{w}^\star}(P \to Q) = 0$.

COMPLETENESS: Suppose $\Gamma \not\vdash_{Rx} P$ for one of the relevant logics under consideration; then by the completeness of the simplified semantics, $\Gamma \not\models_{Sx} P$; so there is an Sx interpretation $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \mathsf{v} \rangle$ s.t. $\mathsf{v}_{\mathsf{g}}(\Gamma) = 1$ but $\mathsf{v}_{\mathsf{g}}(\mathsf{P}) = 0$; so by L4 and L5 there is a 4x interpretation $\langle W, N, \overline{N}, R, \overline{R}, v \rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$; so $\Gamma \not\models_{4x} P$. So if $\Gamma \models_{4x} P$ then $\Gamma \vdash_{Rx} P$.

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