Making Sense of Relevant Semantics (draft)

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Abstract: Involving as it does impossible worlds and the like, the Routley-Meyer worlds semantics for relevant logic has seemed unmotivated to some. I set a version of relevant semantics in a context to make sense of its different elements. Suppose a view which makes room for structured properties — or related entities which combine in arbitrary ways to form structured ones. Then it may seem natural to say *entailment* supervenes upon the structures, so that P entails Q just when part of the condition for BEING P is BEING Q. If P stands in this relation to Q, a result is that there is no possible world where P but not Q, so that P classically entails Q. But the conditions are not equivalent. For all possible worlds, but not all properties, are maximal and consistent. I suggest that relevant semantics is naturally seen as modeling entailment grounded in property structure and makes *sense* insofar as it reflects this fundamental and intuitive notion.

1 Introduction

Against classical logic, relevant logics are typically motivated by syntactical examples, and the sorts of examples used to motivate relevant *vis-à-vis* classical logics are well known. There are the paradoxes of material and

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strict implication, that $p \supset (q \supset p)$, $\neg q \supset (q \supset p)$, $\Box p \rightarrowtail (q \rightarrowtail p)$ and $\Box\neg q \rightarrowtail (q \rightarrowtail p)$. There are the (related) principles that from a contradiction anything follows, and that a theorem follows from anything. At bottom these are because $p \supset q$ is equivalent to $\neg p \vee q$ and it is possible to have $p \supset q$ when p is independent of q. There are also particular cases intended to appeal to intuition. The following *seem* like cases that could be encountered in an introductory logic or philosophy text (these are based on ones from Routley et al., 1982, 6).

> It is not the case that, if this road goes from Chicago to LA then we will reach our destination. So this road goes from Chicago to LA.
>
> If Bob is a rational animal then Bob is human. So if Bob is rational then Bob is human, or if Bob is animal then Bob is human.

Prima facie, in each case, the premise can be true and the conclusion not. Suppose our destination is Boston; then if this road goes from Chicago to LA we will not reach our destination; and similarly if it goes from Chicago to Seattle we will not reach our destination — either way it is not the case that if the road goes from Chicago to LA we will reach our destination; so the premise of the first argument may obtain though the conclusion is false. And it is natural for the Aristotelian at least to deny both disjuncts of the conclusion of the second: gods are rational but not animal and so not human, and rabbits are animal but not rational and so not human; so neither animality nor rationality is sufficient for humanity. If it can be that the premises are true and the conclusion false, the arguments are invalid. But their natural symbolizations, $\neg(c \supset d)$ so c, and $(r \wedge a) \supset h$ so $(r \supset h) \vee (a \supset h)$ are classically valid. Again, this is because $p \supset q$ is equivalent to $\neg p \vee q$. One might think problems go away with strict implication. But this is not obviously right. At least the first argument remains valid if we add that the general path of a road is essential to it, and the second insofar as either rationality or animality is essential to a creature that has it.[1]

[1] So, e.g., $(r \wedge a) \rightarrowtail h$, $a \rightarrowtail \Box a \models_{S5} (r \rightarrowtail h) \vee (a \rightarrowtail h)$. If one is bothered by the appeal to S5, let the additional premise be $\Diamond a \supset \Box a$. This may seem no less acceptable than

One response to such examples is to deny that anything has gone wrong: to deny there is a problem with the paradoxes of implication, with the principles that anything follows from a contradiction and that a theorem follows from anything, or the arguments in the cases. Another response is to grant the invalidity of the ordinary language arguments but to deny that this is a problem for classical logic: in certain circumstances, '$\supset$' and '$⥽$' do not and should not translate 'if'; in these cases, either 'if' should be translated some other way or standard classical and modal logics do not apply; so standard classical and modal logics do not imply the arguments are valid. The first response strikes me as implausible — at least as a response to *all* the examples. The second is not so much a reason to adopt classical logic as it is a reason to adopt a logic (perhaps including classical logic) allowing for a natural translation of 'if' in a broad range of cases. Relevant logics are supposed to be adequate to a range of argumentation to which classical logic is not, and such logics do give the "right" results in many cases.

But philosophers have not embraced relevant logics with open arms. Part of the problem seems to be that as syntactical systems they come across as mere *ad hoc* formulations. It is perhaps particularly difficult to adjudicate intuitions about competing syntactical systems. And intuitions may go both ways. So relevant systems not only block the paradoxes, but also principles many hold dear — notoriously including disjunctive syllogism, the principle that from $P \vee Q$ and $\neg P$ it follows that Q. One might have thought that, as was arguably the case for modal logic, the provision of "worlds" semantics for the logics would have undercut the objection. But the objection remains. The *semantics* has appeared (to some) as a mere *ad hoc* formalism. In particular, the Routley-Meyer worlds semantics has been attacked on the ground that it is a mere formalism, useful from a technical point of view, but otherwise uninteresting (see Copeland, 1979, 1983; van Benthem, 1984; Lewis, 1982).

Consider this picture of the logical project: One begins with some domain of interest; expressions are "true" or "false" on members of that do-

the other, and permits the argument in K. Results are similar if we focus just on worlds where Bob exists. And similarly with counterfactual conditionals of the sort discussed by Stalnaker (1968) and Lewis (1973).

main; an argument is *logically valid* when no member of the domain makes the premises true and the conclusion not. Formal logic begins with some modeling of that domain — perhaps by means of set theory. Formal expressions are true and false on models; an argument is *semantically valid* when no model makes the premises true and the conclusion not. Derivation systems introduce a syntactically defined notion of *formal validity*. Proofs of soundness and completeness demonstrate that the same arguments are formally valid as are semantically valid.

Suppose $\mathcal{D}$ is the original domain, $\mathcal{M}$ is a collection of models, and each P from some class of expressions true or false on members of $\mathcal{D}$ has a formal counterpart P' true or false on members of $\mathcal{M}$. Then if for each $d \in \mathcal{D}$ there is an $m \in \mathcal{M}$ such that a P is true at d iff P' is true at m, the situation is as follows.

$$\boxed{\textit{Logical Validity}} \longleftarrow \boxed{\textit{Semantic Validity}} \longleftrightarrow \boxed{\textit{Formal Validity}}$$

If an argument is logically invalid, then there is some $d \in \mathcal{D}$ that makes the premises true and conclusion not; but then there is an $m \in \mathcal{M}$ that makes formal counterparts of the premises true and conclusion not; so the corresponding formal argument is semantically invalid. Soundness and completeness guarantee that the same arguments are formally valid as are semantically valid. The left-hand arrow does not go both ways, insofar as $\mathcal{D}$ may be subject to constraints to which $\mathcal{M}$ is not, so that not every member of $\mathcal{M}$ has a counterpart in $\mathcal{D}$.

The picture should be familiar from the classical case, where $\mathcal{D}$ is a set of worlds, $\mathcal{M}$ a set of models, and any world $d \in \mathcal{D}$ maps to a model $m \in \mathcal{M}$ such that a P is true at d iff P' is true at m. Thus if an argument is semantically valid — if there is no model where the premises are true and the conclusion not, then there is no world where the premises are true and the conclusion not; so the argument is logically valid. But not every formal model maps in this way to a world; so we have the familiar result that not every argument logically valid is semantically valid. We are thus presented a metaphysical picture underlying the classical notion of logical entailment, one to which the logic is responsible. Other pictures might lead to other logics, as intuitionistic logic, or whatever.

From this perspective, the complaint about relevant logic concerns the left-hand box. There are well-defined relevant derivation systems, with correspondingly clear notions of formal validity. And there are structures, with corresponding notions of semantic validity, such that an argument is formally valid iff it is semantically valid on those structures. The complaint is that the structures do not model any domain of interest. This is no problem, if the object is to study a class of structures by means of its relation to some syntactical derivation system, or if the object is to study a derivation system by means of its relation to some class of structures. But, so far, the project is like one of pure mathematics or, if you will, pure logic. Taken this way, it is no wonder that philosophers have not embraced relevant logic as a tool for reasoning. We make sense of relevant semantics, when we show that its semantic structures are appropriately related to domains of interest. Perhaps there is some domain or another to which the structures are appropriately related. As indicated by examples with which we began, however, relevant logic is supposed to capture something fundamental about entailment; so we want to see how the structures are appropriately related to *that*. *Prima facie*, however, there are significant obstacles to any such attempt:

(i) On the Routley-Meyer semantics, it may be that neither P nor $\neg P$ or both P and $\neg P$ are true at a "world," though disjunction and conjunction work in the usual way. But then one wants to know why these work as usual and the other not. Of course, there is no difficulty about the existence of formal entities at which P and $\neg P$ are each assigned 1 or each assigned 0. In the ordinary case, though, we expect the formal entities to model a domain of interest, and it is because of this relation that formal results, validity and the like, matter. And it is not clear how worlds at which P and $\neg P$ are each assigned 1 or each assigned 0 model any domain of interest. Apart from such an account, it is not suprising that the semantics based on worlds should come across as a mere formalism.

(ii) The semantics includes a ternary relation R on the set W of worlds such that $P \rightarrow Q$ holds at $w \in W$ iff for all $x, y \in W$ if $wRxy$ and P holds at x then Q holds at y. And various more or less unintuitive constraints are placed on R. It is clear enough that the relation and constraints validate principles of the relevant derivation systems. But it is less clear how they correspond to domains we care about. There is a familiar access relation from modal logic for relative possibility; perhaps we understand this; and different accounts of modal facts may suggest one access relation rather than another — with corresponding differences in derivation systems. But the relation from modal logic is binary, and so different from this ternary relation for relevant semantics.

(iii) The semantics includes a unary relation $*$ on worlds such that $w = w^{**}$ and $\neg P$ holds at w iff P does not hold at w^*. But it is not clear why a negation at one world should be determined by what is the case at another. I do not object, as some have, that the $*$ relation is a mere formal trick.[2] Once we accept worlds where expressions are, in some sense, both true and false or neither, there has to be some specification of expressions that are *not false* at w; this is just what w^* does. But, given this, there is room to wonder whether this work is appropriately done by other *worlds*. Suppose neither Pa nor $\neg Pa$ is false at some w (perhaps because a does not exist there). One might resist the supposition that there is therefore a w^* where Pa and $\neg Pa$ are both true. For reasons to allow that neither is false do not automatically translate into reasons to allow that both are true. Star does have certain formal virtues. However, I shall suggest that its formal features disconnect it from domains that might motivate the semantics. Of course, one might somehow constrain the $*$ relation to force an appropriate connection. But then we will want to understand the constraint.

(iv) Finally, as for non-normal modal logic, relevant semantics identifies certain designated "normal" world(s) at which validity is evaluated, and at which special semantic conditions apply. Again, these are at least an effective technical device for validating certain formulas. But it is natural to ask how these worlds and conditions connect to things we care about.

In this paper, I set a version of relevant semantics in a context to make sense of its different elements. The result is not a complete vindication of all things relevant — thus, for example, I eschew the $*$ relation in favor of

[2] See, e.g., Copeland (1979). Early accounts of the relation do leave a lot to be desired. Thus, e.g., we are told that "$*$ is the ordinary reversal operation of turning inside out" (with analogies to the reverse side of a gramophone record and, when $w = w^*$, to a Mobius strip) Routley et al. (1982, 300), Routley (1980, 291). For a more significant account, see Restall (1999).

a four-valued approach, and fault certain standard constraints on R. Still, I shall argue that the relevant approach exhibits a fundamental notion of *entailment*, from the perspective of which classical and other logics may profitably be understood. There are some technical results. In particular, I develop the four-valued approach so as to make contact with standard classical, modal and relevant systems (including the relevant system R), and supply what I take to be an intuitive natural derivation system. But this is meant to serve the main philosophical project. In section 2, I take up the relevant system of first-degree entailment. This puts us in a position to say something about worlds, possible and impossible. On the account I offer, formal worlds model properties, and entailment supervenes on a domain of properties. Negation is correspondingly developed along the lines of the four-valued "American plan." In section 3, I turn to relevant logics beyond the first degree and ternary access. On the account I offer, ternary access captures entailment just insofar as entailment is a two-place relation. In each case, normal worlds constitute an "inner domain" of worlds which effect a transform of results, and may be constrained to take on important features. In the end, I suggest that relevant logic captures an important notion of entailment, and at the same time makes room for ordinary classical logic. Perhaps, then, we all can get along.

2 Properties, Worlds and First-Degree Entailment

I begin with discussion of properties and logical entailment. This leads to the system of first-degree entailment from Anderson and Belnap (1975). The first degree system is presented in two forms: there is a first version, and then a "transform" of the results.

2.1 Properties

Consider a reasonably generic property theory. There are some basic properties akin, perhaps, to fundamental properties of physics. Other properties are constructed from, or supervenient upon, combinations of these. A thing instantiates a conjunction of properties when it instantiates them both, a disjunction when it instantiates one or the other, and a negation when it instantiates a complement given some background class — as a thing is not a pine when it is a cedar, an oak, or the like. Let us suppose that for any set {BEING P, BEING Q...} there are properties BEING P $\wedge$ Q $\wedge$..., and BEING P $\vee$ Q $\vee$.... In addition, each BEING P has a negation, BEING $\neg$P. Notice that for any BEING P there is BEING $\neg$P so that, given arbitrary conjunctions, there is BEING P $\wedge$ $\neg$P. This may seem natural, if we think of properties as independent things subject to arbitrary combinations, and so think of them along Platonistic lines. Of course, some properties formed this way are not *instantiable*, but that does not make them properties any the less.

Once we say this, it is apparent that the approach is incompatible with accounts on which there are no uninstantiated properties (as Armstrong, 1997). Similarly, on our account, BEING P $\wedge$ $\neg$P is one thing, and BEING Q $\wedge$ $\neg$Q is another. So the view is more fine-grained than one on which a property is just the set of its this- and other-worldly instances (as Lewis, 1986). So the view is not entirely generic. At the same time, with some pushing and shoving, our purposes should be served apart from Platonism, and even apart from the appeal to properties. So long as we retain the fine-grained and structured picture, different versions of properties, and even competing objects of philosophical analysis — along the lines of propositions, meanings or the like should do as well. Depending on the objects, though, there may be differences for the *significance* of entailment grounded in the structures — and, as below, I think properties are particularly well-placed to explain why we should care about entailment from the structures for reasoning about the world. With this said, I simply return to the structured properties.

Given a view of this sort, it is reasonable to think that possible worlds where a thing is P are ones where it is Q because of the way BEING P and BEING Q are. Without application to relevant logic (!) Michael Jubien makes this point in a number of places.

> It is a necessary truth that if something is red, then it is colored. But where does this necessity come from?... The necessity comes from the fact that part of what redness does in constituting something as being red is to constitute it as being colored. One way of thinking about this (which may or may not be the way it actu-

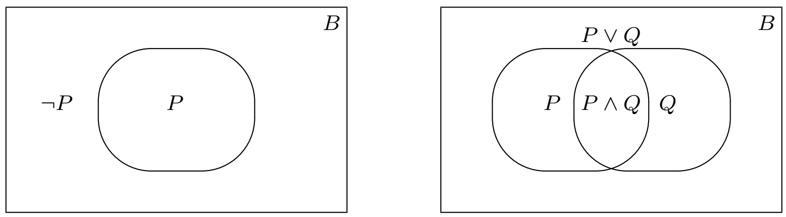
> ally works) is to think of BEING COLORED as an integral part of BEING RED, a subcomplex. At any rate, BEING RED bears a very special relation to BEING COLORED, a relation that "supervenes" on the intrinsic properties of the two complex entities. This relation is such that anything that instantiates the first property is *thereby* compelled to instantiate the second. Let's call this special "intrinsic" relation *entailment*. Jubien (1996, 119-120); see also, Jubien (1993, 111-115), Jubien (2009, 92-94), and Roy (1993, 2012).

Jubien speaks of *property* entailment. I have no problem with this. However, it will be enough if properties are positioned so that they and their relations are fit to function as a ground for entailment.

As Jubien suggests, the point about relations is particularly natural given something like the structured picture from above. Given the nature of BEING P $\vee$ Q with BEING P and BEING Q as parts, a thing which instantiates BEING P instantiates a part of BEING P $\vee$ Q which is sufficient for the instantiation of that property; so a thing which instantiates BEING P thereby instantiates BEING P $\vee$ Q. And a thing which instantiates BEING P $\vee$ Q thereby instantiates BEING P or BEING Q. Similarly a thing which instantiates BEING P $\wedge$ Q thereby instantiates BEING P and BEING Q, and a thing which instantiates BEING P and BEING Q thereby instantiates BEING P $\wedge$ Q. Though it is easy to speak this way, and I shall continue to do so, the point should not be vacuous in the case of uninstantiable properties; the point is rather that the condition for the instantiation of BEING P is included in the very condition which must be met for the instantiation of BEING P $\wedge$ Q. And similarly in other cases.

Say a set w of such properties is *closed* when any $v \subseteq w$ is such that if a thing instantiates the properties in v, then w includes also any properties thereby instantiated by the thing. Corresponding to any such collection is the complex property BEING W of having all the properties in w. Then the instantiation conditions for any BEING P require that it be coinstantiated with some closed BEING W of which it is a constituent. From the above, it is natural to say of a closed BEING W that it has BEING P $\wedge$ Q as a constituent iff it has BEING P and BEING Q as constituents, and BEING P $\vee$ Q as a constituent iff it has BEING P or BEING Q as constituents.

Questions about negation are bound to be controversial. I do not propose or defend a comprehensive theory. Rather, given the larger project, I simply accept what I hope is a sensible picture. Perhaps our initial suggestion on which there are some basic "positive" properties, with negations formed as compliments against a background class, pushes in the direction of a picture along the following lines.



First, as on the left, against some background B of conditions $b_1, b_2 \ldots$ we are given that members of some collection are sufficient for BEING P and of its complement for BEING ¬P. Thus, for example, the background may be a set $S = \{s_1, s_2 \ldots\}$ of shapes, where BEING P includes ones sufficient for being a parallelogram — being square, rectangular and the like, and the negation ones in its complement — being round, triangular, and so forth. Or maybe the background is a set $C = \{c_1, c_2 \ldots\}$ of colors, BEING P ones sufficient for being red, where the negation includes being green, blue and so forth. On this account of negations as compliments, BEING ¬¬P just is BEING P.

If the background for P is the same as for Q, then for the case on the right the background may simply carry over from the prior backgrounds. However, with the examples from above, let the background be the "product" of the prior backgrounds and so, BEING A RED SQUARE, BEING A GREEN SQUARE, BEING A RED CIRCLE, BEING A GREEN CIRCLE, and so forth. Then the condition for BEING P ∧ Q is the intersection of the conditions for BEING P and BEING Q, and the condition for BEING P ∨ Q is their union. And negations appear again naturally as complements. A thing that satisfies the compliment of BEING P ∨ Q thereby satisfies the compliments of BEING P and BEING Q. So a thing that instantiates BEING ¬(P ∨ Q) thereby instantiates BEING ¬P and BEING ¬Q. And similarly the other way around. And similarly, reasoning from the picture, with conjunction.[3]

Thus, where BEING W is closed, and so includes as a constituent any property a thing thereby instantiates when it has BEING W, we arrive at the following.

(neg) BEING W has BEING ¬¬P as a constituent iff it has BEING P as a constituent.

(cnj) BEING W has BEING P ∧ Q as a constituent iff it has BEING P and BEING Q as constituents; and BEING ¬(P ∧ Q) as a constituent iff it has BEING ¬P or BEING ¬Q as constituents.

(dsj) BEING W has BEING P ∨ Q as a constituent iff it has BEING P or BEING Q as constituents; and BEING ¬(P ∨ Q) as a constituent iff it has BEING ¬P and BEING ¬Q as constituents.

So long as we are interested in sentential conditions, we shall be particularly interested in properties that are completed *states* of worlds as, BEING SUCH THAT BOB IS HAPPY, or BEING SUCH THAT BOB IS TALLER THAN SUE. These properties behave according to these conditions no less than any others.

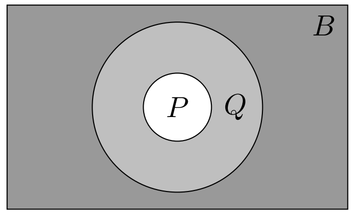
The suggestion has been that entailment supervenes on property structures. It is natural to express this point as follows,

LV *P logically entails Q* just in case no closed BEING W has BEING P as a constituent without BEING Q as a constituent.

Where entailment supervenes on property structures, we track entailment over the domain of closed properties. Put roughly, P logically entails Q just when anything that instantiates BEING P thereby instantiates BEING Q. Observe that, with the account of negations as compliments, this picture leads us to expect that if P logically entails Q, then $\neg Q$ logically entails $\neg P$. Suppose in instantiating BEING P a thing thereby instantiates BEING

[3] Relevant logic is not dependent on a simple picture of negations as complements as above. Thus it is compatible with accounts on which BEING P and BEING ¬P are not exclusive and not exhaustive — so that things may be both P and $\neg P$ and neither. At the same time, as below, neither are such accounts required for relevant logic. For relevant logic on a more general account of negation as "otherness" see Routley and Routley (1985).

Q; then the conditions sufficient for BEING P are a subset of those sufficient for BEING Q.



So, with the above picture, the complement of BEING P is the entire shaded area and of BEING Q the outer; so the compliment of BEING Q is a subset of the compliment of BEING P; and in instantiating BEING ¬Q a thing thereby instantiates BEING ¬P.

So far, all this may seem the classical picture, or a good part of it, cast in terms of a reasonably traditional picture of structured properties. But, of course, part of the classical picture is missing. One might suggest that, given the basic structure of negations as compliments, closed properties are subject also to the constraint that BEING ¬P is a constituent of a closed property if and only if BEING P is not. With this, the condition from (neg) is redundant, and the double conditions from (cnj) and (dsj) collapse to the usual classical form. But this proposed condition goes beyond closure as requiring just what a thing thereby instantiates when it instantiates a property. A thing may instantiate some BEING Q without thereby instantiating either BEING P or BEING ¬P. So it may be that a closed property has neither as a constituent, and so fails to have BEING P ∨ ¬P as a constituent. And similarly a closed property may include both BEING P and BEING ¬P and so BEING P ∧ ¬P. On our fine-grained and structured picture, there is nothing mysterious about incomplete and inconsistent properties. Thus the proposed additional constraint does more than merely unpack conditions a thing thereby does or does not satisfy when it instantiates BEING P or BEING ¬P. And so far as conditions on the intrinsic features of properties go, conjunction and disjunction work classically but negation does not, so that we are left with (neg), (dsj) and (cnj) for our account of closure.

Of course, nothing prevents applying the extra condition for the classical result. Say a closed BEING W is *world-like* when for any BEING P it has either BEING P or BEING ¬P as constituents but not both. Then, CV *P classically entails Q* just in case no closed, world-like BEING W has BEING P as a constituent without BEING Q as a constituent.

Clearly logical validity guarantees classical validity. But not the other way around, for the world-like closed properties are a subset of all closed properties. So, on our account, LV does not express the classical notion of validity. As we shall see, however, LV is consistent with the relevant approach — and it may seem useful to separate the logical and classical notions of entailment.

There may be a direct intuition that this notion of property inclusion captures an important notion of entailment. But there are theoretical motivations as well. As suggested in the quotation from Jubien, philosophers have often looked, in one way or another, to the internal structure of properties (propositions, meanings, or whatever) for explanation of modal facts — and so to the internal structure of "properties" for explanation of the way possible worlds are. In one mode, even Quine (1980, 1976), allows meaning and analyticity as a ground of modal claims. I defend a related but realist view of the ground for modality (Roy, 1993, 2000, 2012). At any rate, the basic idea seems natural. Insofar as entailment is defined classically over the universe of possible worlds, property structures are thus supplied as a ground for classical entailment. But, on an intuitive picture of the properties (propositions, meanings, or the like) on which they permit arbitrary combinations, this cannot be the whole story. Rather, the classical notion of entailment emerges from the structured picture, only when properties are limited to ones that are world-like. We thus isolate a pair of metaphysical notions underlying the classical account, and offer an account of how they combine to result in the standard notion.

2.2 Basic System: *FA*

Anderson and Belnap's system of first-degree entailment is a fragment of standard relevant logics, including B, DW, TW, RW and R. In this section, I develop the first-degree semantics from Dunn (1976). I begin with statement of the details, and return to the question of significance.

Begin with a standard language for sentential logic with propositional parameters p_1, $p_2 \ldots$, operators $\neg$, $\wedge$, $\vee$ and $\supset$, and sentences formed in the usual way. An interpretation is a function v that assigns to each parameter some subset of $\{1, 0\}$. So $v(p)$ is ϕ, $\{1\}$, $\{0\}$, or $\{1, 0\}$. For now, leave this uninterpreted. For complex expressions,

TFA ($\neg$) $1 \in v(\neg P)$ iff $0 \in v(P)$
$0 \in v(\neg P)$ iff $1 \in v(P)$
($\wedge$) $1 \in v(P \wedge Q)$ iff $1 \in v(P)$ and $1 \in v(Q)$
$0 \in v(P \wedge Q)$ iff $0 \in v(P)$ or $0 \in v(Q)$
($\vee$) $1 \in v(P \vee Q)$ iff $1 \in v(P)$ or $1 \in v(Q)$
$0 \in v(P \vee Q)$ iff $0 \in v(P)$ and $0 \in v(Q)$
($\supset$) $1 \in v(P \supset Q)$ iff $0 \in v(P)$ or $1 \in v(Q)$
$0 \in v(P \supset Q)$ iff $1 \in v(P)$ and $0 \in v(Q)$

For a set Γ of formulas, $1 \in v(\Gamma)$ iff $1 \in v(P)$ for each $P \in \Gamma$. Then,

VFA $\Gamma \models_{FA} P$ iff there is no *FA* interpretation v such that $1 \in v(\Gamma)$ but $1 \notin v(P)$.

If interpretations are constrained so that $1 \in v(p)$ iff $0 \notin v(p)$, then $v(p) = \{0\}$ or $v(p) = \{1\}$, and these conditions are entirely classical. Without such constraint, with four-valued interpretations, the result is relevant in the sense that if $P \models_{FA} Q$, then P and Q have some parameter in common.[4] It is worth noting that there is no interpretation v such that $1 \in v(P)$ and $1 \notin v(Q)$ just in case there is no interpretation v such that $0 \notin v(P)$ and $0 \in v(Q)$. Thus corresponding to the "positive" condition for validity is an equivalent "negative" condition.[5]

[4]To see this, suppose P and Q have no parameter in common; then there is an interpretation v that assigns $\{1, 0\}$ to each p in P and ϕ to each q in Q; then, by an easy induction, $v(P) = \{1, 0\}$ and $v(Q) = \phi$; so $1 \in v(P)$ but $1 \notin v(Q)$; and since there is such an interpretation, $P \not\models_{FA} Q$.

[5] The point is from Dunn (1976, 165), cf. Dunn (2000). To see this, note that any v for *FA* has a "dual" v^* such that $1 \in v^*(p)$ iff $0 \notin v(p)$ and $0 \in v^*(p)$ iff $1 \notin v(p)$. Then by a simple induction, $1 \in v^*(P)$ iff $0 \notin v(P)$ and $0 \in v^*(P)$ iff $1 \notin v(P)$. But if $P \not\models_{FA} Q$, there is a v such that $1 \in v(P)$ but $1 \notin v(Q)$; so $0 \notin v^*(P)$ but $0 \in v^*(Q)$; so if an argument from P to Q fails the "positive" condition for validity, it fails the "negative" one as well. And similarly in the other direction. (Another way to see the point is to note that derivation rules from below are symmetric insofar as switching each $/P/$ for $\backslash P\backslash$ in a derivation results in a derivation, so that there is an *FA* derivation from P to Q iff there is an *FA* derivation from $\overline{P}$ to $\overline{Q}$.)

There are different ways to approach derivations. What follows develops a strategy from Woodruff (1970). Introduce expressions of the sort P and $\overline{P}$. Intuitively, $\overline{P}$ indicates that P is *not false*. Let $/P/$ and $\backslash P\backslash$ represent either P or $\overline{P}$ where what is represented is constant in a given context but $/P/$ and $\backslash P\backslash$ are opposite, and similarly for $/\!/P/\!/$ and $\backslash\!\backslash P\backslash\!\backslash$ — though $/P/$ and $/\!/P/\!/$ need not be the same. Rules are on the pattern of I- and E-rules for standard Fitch systems, as in Bergmann et al. (2004) and Roy (draftC).

$$\begin{array}{l|l} \textbf{R} & /P/ \\ & \\ & /P/ \end{array} \qquad \begin{array}{l|l} \neg\textbf{I} & \begin{array}{|l} /P/ \\ \hline \\ /\!/Q/\!/ \\ \backslash\!\backslash\neg Q\backslash\!\backslash \end{array} \\ & \backslash\neg P\backslash \end{array} \qquad \begin{array}{l|l} \neg\textbf{E} & \begin{array}{|l} /\neg P/ \\ \hline \\ /\!/Q/\!/ \\ \backslash\!\backslash\neg Q\backslash\!\backslash \end{array} \\ & \backslash P\backslash \end{array}$$

$$\begin{array}{l|l} \wedge\textbf{I} & /P/ \\ & /Q/ \\ & \\ & /P \wedge Q/ \end{array} \qquad \begin{array}{l|l} \wedge\textbf{E} & /P \wedge Q/ \\ & \\ & /P/ \end{array} \qquad \begin{array}{l|l} \wedge\textbf{E} & /P \wedge Q/ \\ & \\ & /Q/ \end{array}$$

$$\begin{array}{l|l} \vee\textbf{I} & /P/ \\ & \\ & /P \vee Q/ \end{array} \qquad \begin{array}{l|l} \vee\textbf{I} & /P/ \\ & \\ & /Q \vee P/ \end{array} \qquad \begin{array}{l|l} \vee\textbf{E} & /P \vee Q/ \\ & \begin{array}{|l} /P/ \\ \hline \\ /\!/R/\!/ \end{array} \\ & \begin{array}{|l} /Q/ \\ \hline \\ /\!/R/\!/ \end{array} \\ & /\!/R/\!/ \end{array}$$

$$\begin{array}{l|l} \supset\textbf{I} & \begin{array}{|l} /P/ \\ \hline \\ \backslash Q\backslash \end{array} \\ & \backslash P \supset Q\backslash \end{array} \qquad \begin{array}{l|l} \supset\textbf{E} & \backslash P \supset Q\backslash \\ & /P/ \\ & \\ & \backslash Q\backslash \end{array}$$

Where the members of Γ and P are without overlines, $\Gamma \vdash_{FA} P$ iff there is a derivation of P from the members of Γ. As an example, $(A \wedge B) \supset C \vdash_{FA} (A \wedge \neg C) \supset \neg B$.

1	$(A \wedge B) \supset C$	prem
2	$\quad \overline{A \wedge \neg C}$	A ($\supset$I)
3	$\quad\quad \overline{B}$	A ($\neg$I)
4	$\quad\quad \overline{A}$	2 $\wedge$E
5	$\quad\quad \overline{A \wedge B}$	3,4 $\wedge$I
6	$\quad\quad C$	1,5 $\supset$E
7	$\quad\quad \overline{\neg C}$	2 $\wedge$E
8	$\quad \neg B$	3-7 $\neg$I
9	$(A \wedge \neg C) \supset \neg B$	2-8 $\supset$I

The derivation system is sound and complete, $\Gamma \vdash_{FA} P$ iff $\Gamma \models_{FA} P$, and remains sound and complete for classical logic if interpretations are constrained so that $1 \in v(p)$ iff $0 \notin v(p)$, and we add a rule moving from $/P/$ to $\backslash P\backslash$. For further examples, with demonstrations of soundness and completeness, see §6 of Roy (2006).

Say an interpretation v is a "world." Then worlds are structures at which an expression P may be assigned both 1 and 0 or neither. If worlds model some domain of concern, it is natural to ask what sort of thing members of this domain may be. On response is that they are, well, *worlds* — of the sort that can be and are actual — where $1 \in v(P)$ iff P is true at a world, and $0 \in v(P)$ iff P is false. So P may be both true and false at a world or neither. That P may be both true and false seems suggested by Priest (2001, §7.6 - 7.9); compare Priest (1987, §9.7) for discussion of related issues. Priest has many interesting things to say — like Lewis on worlds, he makes interesting arguments for conclusions that elicit the "incredulous stare." However, even if he is right and there are true contradictions at worlds of the sort that may be actual, I do not think the relevant notion of logical validity is to be understood this way. For it is not Priest's view that for any old contradiction, there is a possible world where it is true. He argues for true contradictions under certain conditions — as under conditions of self-reference. But such considerations are not reasons to think that, say, there are possible worlds where he both is and is not over six feet tall. So an argument from the premise that he is and is not over six feet tall to the conclusion that kangaroos fly is logically valid on the domain of such worlds, just because there is no possible world where he is and is not over six feet tall.[6] Something similar might be said for expressions that are neither true nor false: Perhaps considerations about failed reference or the like induce us to allow that some expressions may be neither true nor false at a world. But such considerations will not obviously apply to arbitrary expressions, and so will not generally underwrite the relevant notion of entailment.

Other authors appeal to entities which may, in some sense, uncontroversially accommodate arbitrary inconsistencies and gaps. Thus, for example, one might appeal to *theories*, to *information*, to *sets of propositions*, to *properties* or the like (see Belnap, 1977; Restall, 1996a; Mares, 1996, 2004). Depending on one's ontology, some of these may come to very much the same thing. But there are twin pressures: We need to see not only how the entities accommodate inconsistencies and gaps, but also whether and how they are appropriately constrained. As above, I think interpretations naturally model closed properties.

It is, in fact, a very short step from logical validity as described above, to the semantic conditions for *FA*. For some closed property BEING W set $1 \in v(P)$ just in case BEING P is a constituent of BEING W and $0 \in v(P)$ just in case BEING ¬P is a constituent of BEING W. Consistent with constraints on BEING W, it may be that BEING P is a constituent of BEING W, that BEING ¬P is a constituent of BEING W, that both are constituents, or that neither are constituents. So there are the options that $v(P) = \phi$, $\{1\}$, $\{0\}$, and $\{1, 0\}$. Turning to the conditions for TFA: By the modeling, $1 \in v(P \wedge Q)$ iff BEING P ∧ Q is a constituent of BEING W; by (cnj) iff BEING P and BEING Q are constituents of BEING W; by the modeling, iff $1 \in v(P)$ and $1 \in v(Q)$. Similarly, by the modeling, $0 \in v(P \wedge Q)$ iff BEING ¬(P ∧ Q) is a constituent of BEING W; by (cnj) iff BEING ¬P or BEING ¬Q are constituents of BEING W; by the modeling, iff $0 \in v(P)$ or $0 \in v(Q)$. And similarly for (∨). For negation, by the modeling, $1 \in v(\neg P)$ iff BEING ¬P is a constituent of BEING W; by the modeling again, iff $0 \in v(P)$. And by the modeling, $0 \in v(\neg P)$ iff BEING ¬¬P is a constituent of BEING W; by (neg) iff BEING P is a constituent of BEING W; by the modeling iff $1 \in v(P)$. So, supposing ⊃ abbreviates an expression in ¬ and ∧ or ∨, we recover the conditions from TFA. One might

[6] In conversation, he seems inclined either to allow that such arguments are in fact valid, or to appeal to *impossible* worlds — ones not themselves motivated by the considerations from self-reference and the like. On the latter, see Priest (2001, §9.7).

object that it is *too* short a step from the theory of properties to the semantic conditions — the theory of properties is somehow engineered to make the semantics go. But I think this is a mistake. Rather, the standard conditions have, presumably, been intended all along to characterize these properties.

So we are left with the following picture: There is a domain $\mathcal{D}$ of closed properties. P logically entails Q just in case there is no BEING W $\in \mathcal{D}$ such that BEING P is a constituent of BEING W but BEING Q is not. These conditions are modeled by the semantics for *FA*. And the derivation system for *FA* is adequate to this semantics. So we seem to have all the elements to represent the First Degree system as reflecting an *account* of entailment as grounded in the nature of properties.

2.3 Results Revised: *FB*

Actually, *FA* is not quite *FDE* as described by Anderson and Belnap. *FDE* has an operator $\rightarrow$, where formation rules for other operators are as usual, and if A and B are sentences without any instance of $\rightarrow$, then $A \rightarrow B$ is a sentence. *FDE* is then developed as an axiom system with derivations and theorems in the usual way. But *FA* has no theorems at all.[7] So the systems are not equivalent. Rather, as Dunn shows, they are related so that, $P \models_{FA} Q$ iff $\vdash_{FDE} P \rightarrow Q$. But we can convert our valid arguments into theorems as for *FDE*. This comes to a sort of transformation, one that will introduce normal worlds for the first time. Again, I begin with details, and return to the question of significance.

Let the language have parameters $p_1, p_2 \ldots$ and operators, $\neg$, $\wedge$, $\vee$, $\supset$ and $\rightarrow$. Each propositional parameter is a *formula*; if P and Q are formulas, so are $\neg P$, $P \wedge Q$, $P \vee Q$, $P \supset Q$, and $P \rightarrow Q$. Then, for this first-degree system, a *sentence* is any formula where no instance of $\rightarrow$ appears in the scope of another. An interpretation is $\langle W, N, \overline{N}, v\rangle$ where W is a set of worlds; $N, \overline{N} \subseteq W$ are sets of *normal* worlds for truth and non-falsity respectively; and v is a valuation which assigns to each parameter some subset of $\{1, 0\}$ at each $w \in W$. This time, it will be convenient to specify a valuation h directly for expressions of the sort that have so far appeared

[7] To see this, consider an interpretation v such that for any p, $v(p) = \phi$; then by an easy induction, $v(P) = \phi$; and since there is such an interpretation, $\not\models_{FA} P$.

in derivations. Say $/P/$ *holds* at w just in case $h_w(/P/) = 1$ and otherwise *fails*.

HFB (B) $h_w(p) = 1$ iff $1 \in v_w(p)$; $h_w(\overline{p}) = 1$ iff $0 \notin v_w(p)$

($\neg$) $h_w(/\neg P/) = 1$ iff $h_w(\backslash P\backslash) = 0$

($\wedge$) $h_w(/P \wedge Q/) = 1$ iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$

($\vee$) $h_w(/P \vee Q/) = 1$ iff $h_w(/P/) = 1$ or $h_w(/Q/) = 1$

($\supset$) $h_w(/P \supset Q/) = 1$ iff $h_w(\backslash P\backslash) = 0$ or $h_w(/Q/) = 1$

($\rightarrow$) for $w \in /N/$, $h_w(/P \rightarrow Q/) = 1$ iff there is no $x \in W$ such that $h_x(/\!/P/\!/) = 1$ and $h_x(/\!/Q/\!/) = 0$; otherwise $h_w(/P \rightarrow Q/)$ is arbitrary

Conditions for the classical operators $\neg$, $\wedge$, $\vee$ and $\supset$ parallel ones from before. And from the perspective of a world in $/N/$, the condition for $/P \rightarrow Q/$ "looks" just like the double condition for validity in *FA* — the conditional holds when there is no world where $/\!/P/\!/$ holds but $/\!/Q/\!/$ does not. Where the members of Γ and P are without overlines, $h_w(\Gamma) = 1$ iff $h_w(A) = 1$ for each $A \in \Gamma$; and,

VFB $\Gamma \models_{FB} P$ iff there is no *FB* interpretation $\langle W, N, \overline{N}, v\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

Again, if interpretations are constrained so that $1 \in v_w(p)$ iff $0 \notin v_w(p)$, and N is set equal to $\overline{N}$ then (given that validity is evaluated in normal worlds), results are as in classical modal logic, with $\rightarrow$ the strict conditional.

For a derivation system, where s is any integer, let $/P/_s$ be a *subscripted formula*. Intuitively, subscripts indicate worlds, where $/P/_s$ when $/P/$ holds at world s. $/n/[s]$ indicates that world s is in $/N/$. Rules for $\neg$, $\wedge$, $\vee$ and $\supset$ are a natural development from *FA*.

$$\begin{array}{r|l} \textbf{R} & /P/_s \\ & \\ & /P/_s \end{array} \qquad \begin{array}{r|l} \neg\textbf{I} & \begin{array}{|l} /P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash\neg P\backslash_s \end{array} \qquad \begin{array}{r|l} \neg\textbf{E} & \begin{array}{|l} /\neg P/_s \\ \hline \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash P\backslash_s \end{array}$$

$$\wedge\mathbf{I}\ \begin{array}{|l} /P/_s \\ /Q/_s \\ \\ /P \wedge Q/_s \end{array} \qquad \wedge\mathbf{E}\ \begin{array}{|l} /P \wedge Q/_s \\ \\ /P/_s \end{array} \qquad \wedge\mathbf{E}\ \begin{array}{|l} /P \wedge Q/_s \\ \\ /Q/_s \end{array}$$

$$\vee\mathbf{I}\ \begin{array}{|l} /P/_s \\ \\ /P \vee Q/_s \end{array} \qquad \vee\mathbf{I}\ \begin{array}{|l} /P/_s \\ \\ /Q \vee P/_s \end{array} \qquad \vee\mathbf{E}\ \begin{array}{|l} /P \vee Q/_s \\ \begin{array}{|l} /P/_s \\ \hline \\ /\!\!/R/\!\!/_t \end{array} \\ \begin{array}{|l} /Q/_s \\ \hline \\ /\!\!/R/\!\!/_t \end{array} \\ /\!\!/R/\!\!/_t \end{array}$$

$$\supset\mathbf{I}\ \begin{array}{|l} \begin{array}{|l} /P/_s \\ \hline \\ \backslash Q\backslash_s \end{array} \\ \backslash P \supset Q\backslash_s \end{array} \qquad \supset\mathbf{E}\ \begin{array}{|l} \backslash P \supset Q\backslash_s \\ /P/_s \\ \\ \backslash Q\backslash_s \end{array}$$

$$\mathbf{NI}\ \begin{array}{|l} \\ \\ n[0] \end{array} \qquad \to\mathbf{I}\ \begin{array}{|l} /n/[s] \\ \begin{array}{|l} /\!\!/P/\!\!/_t \\ \hline \\ /\!\!/Q/\!\!/_t \end{array} \\ /P \to Q/_s \end{array} \qquad \to\mathbf{E}\ \begin{array}{|l} /n/[s] \\ /P \to Q/_s \\ /\!\!/P/\!\!/_t \\ \\ /\!\!/Q/\!\!/_t \end{array} \qquad \mathbf{C}\ \begin{array}{|l} /n/[s] \\ /\!\!/P/\!\!/_s \\ \\ \backslash\!\!\backslash P\backslash\!\!\backslash_s \end{array} \quad \begin{array}{|l} /n/[s] \\ \\ \\ \backslash n\backslash[s] \end{array}$$

where t does not appear in any undischarged premise or assumption

Any subscript is 0 or introduced as t in an assumption for $\to$I. For this first-degree system, derivations are required to respect formation rules, so no instance of $\to$ appears in the scope of another. The basic *FB* system takes all the rules except (C). Where the members of Γ and P are without subscripts or overlines, let Γ_0 be those same expressions, each with subscript 0. Then $\Gamma \vdash_{FB} P$ iff there is an *FB* derivation of P_0 from the members of Γ_0. Corresponding to the above example for *FA*, $\vdash_{FB} [(A \wedge B) \supset C] \to [(A \wedge \neg C) \supset \neg B]$.

1	$n[0]$	NI
2	$\quad [(A \wedge B) \supset C]_1$	A ($\to$I)
3	$\quad\quad (\overline{A \wedge \neg C})_1$	A ($\supset$I)
4	$\quad\quad\quad \overline{B}_1$	A ($\neg$I)
5	$\quad\quad\quad \overline{A}_1$	3 $\wedge$E
6	$\quad\quad\quad (\overline{A \wedge B})_1$	4,5 $\wedge$I
7	$\quad\quad\quad C_1$	2,6 $\supset$E
8	$\quad\quad\quad \overline{\neg C}_1$	3 $\wedge$E
9	$\quad\quad \neg B_1$	4-8 $\neg$I
10	$\quad [(A \wedge \neg C) \supset \neg B]_1$	3-9 $\supset$I
11	$([(A \wedge B) \supset C] \to [(A \wedge \neg C) \supset \neg B])_0$	1,2-10 $\to$I

The derivation system is sound and complete, so that $\Gamma \vdash_{FB} P$ iff $\Gamma \models_{FB} P$ (Roy, draftB, §7). And results from *FA* are converted into theorems so that $P \vdash_{FA} Q$ iff $\vdash_{FB} P \to Q$ [A1].[8] We therefore recover Anderson and Belnap's *FDE* in the sense that, $\vdash_{FB} P \to Q$ iff $\vdash_{FDE} P \to Q$. More generally, we recover the complete first degree fragment of relevant logics including *B*, *DW*, *TW* and *RW* — but not, without further constraint, *R*; thus, where the members of Γ and P are sentences in the current language, and *Sx* is one of these standard relevant logics, $\Gamma \vdash_{FB} P$ iff $\Gamma \vdash_{Sx} P$ [A2].

So entailments in *FA* are converted to theorems in *FB*. So theorems carry over motivations from before, and to this extent motivate current machinery. But the notion of conversion has application only to theorems — and it is left open what to make of normal worlds, with semantic and formal validity more generally in *FB*. By way of response, observe that, at worlds in N where validity is measured, the condition for truth of conditionals looks like the condition for validity in *FA*, but the condition for their non-falsity is left wide open. Conditions for truth and non-falsity of conditionals come together with the requirement that $N = \overline{N}$. And normal worlds are given a consistent interpretation as *world-like* by imposing as a full classical constraint,

CL (i) $w \in N$ iff $w \in \overline{N}$

(ii) for any $w \in N$, $0 \notin v_w(p)$ iff $1 \in v_w(p)$.

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[8] References in square brackets are to the Appendix.

Then, if for a system *FBc* we accept rule (C), without change to theorems of the sort $P \rightarrow Q$, we recover ordinary classical logic: for classical forms, $\Gamma \vdash_{FBc} P$ iff $\Gamma \vdash_{CL} P$. In fact, nothing stops us augmenting the language to include $\Box$ (and $\Diamond$), moving to interpretations $\langle W, M, N, \overline{N}, v\rangle$ where $M \subseteq W^2$ is a modal access relation which might be restricted in the usual ways. Then, where modal access from a normal world is restricted just to other normal worlds, still without change to theorems of the sort $P \rightarrow Q$, we recover ordinary modal logic as well.[9] This sacrifices a complete match to the first-degree fragment of ordinary relevant systems. However, in this way, normal worlds are interpreted. And the classical notions coexist with relevant entailment, insofar as the classical notions depend on just a subset of all the worlds, where entailment applies to them all.

The resultant picture has attractions from both the classical and relevant perspectives. Say *FBc* includes modal notions and the classical constraint CL. Then all of ordinary classical and modal logic remains. There is an *additional* operator $\rightarrow$ to represent entailment in the sense of LV. And it may be useful to have some such notion. Perhaps this is immediate from our initial examples. But similarly for various substantive topics in philosophy. So for example, in one place, a proposed analysis, "if it is within S's power to bring it about that P and if that P entails that Q, then it is within S's power to bring it about that Q" is summarily rejected: this principle "is obviously false: Neil Armstrong's being the first human to walk on the moon entails that 2 + 2 = 4, but neither Armstrong nor anyone else has ever had the power to bring it about that this arithmetical proposition is true" (Hasker, 1989, 108). The proposed analysis may very well be false. But, from the present perspective, this move seems too fast. For nothing prevents $\Box Q$ without $P \rightarrow Q$, as $\models_{FBc} \Box(q \vee \neg q)$ but $\not\models_{FBc} p \rightarrow \Box(q \vee \neg q)$ — for the one depends just on normal worlds, and the other not. Examples could be multiplied. Insofar as philosophers are generally interested in *analysis* of properties, as analyses of supervenience, truthmaking, intrinsic properties and the like, it seems natural to employ a notion of entailment no less fine-

[9] So, we require (MC) if $w \in /N/$ and wMx, then $x \in /N/$, and augment HFB to include, ($\Box$) $h_w(/\Box P/) = 1$ iff there is no $x \in W$ such that wMx and $h_x(/P/) = 0$. Optionally, we might require *reflexivity*, (ρ) for all x, xMx; *symmetry*, (σ) for all x, y if xMy then yMx; *transitivity* (τ) for all x, y, z if xMy and yMz then xMz; or the like.

grained than the properties themselves (whatever their ontological account). And given such a notion of entailment, there is no reason to reject analyses as above.[10]

And from the other side, *FBc* still has its relevant conditional. Once inconsistent and incomplete worlds are allowed, it is hard to see why one might not be interested in a class of them where the classical constraints apply — so that members model "world-like" properties, and may be interesting for precisely that reason. In these systems with theorems, logical entailment is indicated by $\rightarrow$, not $\models$. Thus $\models_{FBc}$ indicates just what obtains at classically characterized worlds — though this may itself depend on ones not so constrained. So, turning to a couple of the cases with which we began, $\not\models_{FBc} (p \wedge \neg p) \rightarrow q$; for $p \wedge \neg p$ might hold at a non-normal world, though q does not. It remains that $p \wedge \neg p \models_{FBc} q$; but this merely reflects a feature of normal worlds; given the classical constraint, vacuously, q holds at every normal world where $p \wedge \neg p$ holds. Corresponding to this case is the standard relevant treatment of disjunctive syllogism: $\not\models_{FBc} [(q \vee p) \wedge \neg p] \rightarrow q$. For the full range of worlds includes ones where p and $\neg p$ hold but q does not. At the same time, $q \vee p, \neg p \models_{FBc} q$. But this obtains only given the additional constraint that not both p and $\neg p$ hold at normal worlds. So classical logic reappears, but subject to its own special constraints, and seems thus *explained* from the relevant point of view, without impugning or removing the relevant account of entailment.[11]

3 Ternary Access and Logics Beyond First Degree

I begin this section with a concern that arises for our modeling when a property may have entailment properties as constituents, and then consider ternary access as a response. This leads to a pair of relevant systems beyond

[10] Observe that, though the logic is paraconsistent in the sense that it accommodates inconsistent premises, the applications are not to a special domain (though this is possible) but to relations in which we may be specially interested. For a positive application of relevant logic to the analysis of intrinsic properties see (Dunn, 1987, 1990; Dunn and Restall, 2002, §5.6). Restall (1996b) suggests an application to truthmaking.

[11] There is no unanimity about merging classical and relevant principles. Compare (Restall, 1999, §4) and (Belnap and Dunn, 1981, §5) for positive and then negative assessments.

the first degree. Again, there is a first version, and a transform of the results. Systems from this section are based on Routley (1984) and Roy (draftA).

3.1 Following

Nothing prevents dropping constraints on formation rules for $\rightarrow$, and corresponding constraints on semantics and derivations. But the result is not relevant logic. If $\rightarrow$I applies at arbitrary worlds, we might reason as follows to show, say, $p \rightarrow (q \rightarrow q)$.

1	p_1	A ($\rightarrow$I)
2	q_2	A ($\rightarrow$I)
3	q_2	2 R
4	$(q \rightarrow q)_1$	2-3 $\rightarrow$I
5	$[p \rightarrow (q \rightarrow q)]_0$	1-4 $\rightarrow$I

So $A \rightarrow B$ where A and B have no parameter in common. But this is to be expected. We have thought of properties as subject to arbitrary combinations, and entailment as defined over a domain of closed properties. We now have *entailment properties* as constituents of closed properties. But we do not so far have the capacity to model arbitrary entailment properties as constituents. Indeed, supposing the substantive condition from HFB($\rightarrow$) is extended to arbitrary worlds, BEING (SUCH THAT) P $\rightarrow$ P is a constituent of any closed property represented by our models. There is no world where $/\!/P/\!/$ both holds and does not; so $/P \rightarrow P/$ holds at every world. So we represent BEING P $\rightarrow$ P as a constituent of any closed property. But given arbitrary combinations, we expect closed properties with neither BEING P $\rightarrow$ P nor BEING $\neg$(P $\rightarrow$ P) as constituents, and closed properties with both. Where entailment properties are constituents of closed properties, our approach to entailment requires some means of modeling them all.

The Routley-Meyer semantics for relevant logics enables a model of the full range of entailment properties by ternary access. Where a binary access relation gives a world access to individual worlds, a ternary access relation gives a world access to pairs of others. The basic strategy is to model entailment by access to world-pairs, where the condition for $P \rightarrow Q$ requires that there is no accessible world-pair $\langle x, y\rangle$ such that P holds at x but Q

fails at y. Then, though there is no x where P holds and does not hold, a world w might have access to a pair $\langle x, y\rangle$ such that P holds at x but not at y — so that $P \to P$ fails at w. And we are similarly positioned to represent arbitrary entailment properties as constituents of closed properties. So far, then, this seems to fulfill a requirement for the property-based picture of entailment.

Insofar as entailment and the associated notion of one property being *included* in another are two-place relations, the strategy may seem natural. A binary access relation gives a world access to a set of individuals. A ternary access relation gives a world access to a set of pairs. But entailment and inclusion are binary relations, and the standard way to represent a binary relation is by a set of pairs. Since we want to represent just such a relation, ternary access is natural in this context. Different notions of necessity, alethic, deontic, epistemic and whatever are represented as generalizations over worlds by a modal access relation. And similarly, there may be a class of binary notions naturally represented as generalizations over world-pairs.

So, for example, suppose worlds are divided into a universe of temporal "slices." Say slice b *t-follows* slice a just in case a is prior to b in the temporal ordering of the world from which a and b are carved. Then we might say Q *necessarily t-follows* P just in case there is no $\langle a, b\rangle$ such that b t-follows a, where P is true at a, but Q is not true at b. Perhaps this is an interesting notion about which we could reason in different ways — and something of the sort might arise naturally in a logic with both modal and temporal operators. Given the temporal picture, one might impose restrictions on access to pairs corresponding to different structures for the "t-follows" relation. But the point here is just that necessary following appears as a generalization over slice pairs. Or, more fancifully, suppose world y "follows" x just when David Lewis prefers y to x. Then we might say Q is *unambiguously preferable* to P at w just in case there are no x, y such that $wRxy$ where P is true at x but Q is not true at y. And there might be a corresponding logic of "Lewisian preferences."

But entailment is a binary notion similarly structured. One might interpret the relevant access relation by means that presuppose already something like the required notion of entailment. So Meyer (2004) has $wRxy$ just in case w is a domain of laws; x inputs to the laws; and y outputs given the inputs; arrow statements are syntactical expressions of the laws (Beall et al., 2012, is a useful survey of approaches to the access relation). This will not illuminate entailment for those who do not already understand it. But it is not required. The three place relation gives worlds access to pairs for which we already have a substantive account in terms of inclusion. Entailment then arises as a generalization over entities standing in this relation.

Necessary t-following and unambiguous preference may seem relatively intuitive insofar as, from a Humean perspective at least, there are no *a-priori* restrictions on options for accessible world-pairs. Lewis might prefer any world to another; and slice a might be followed by any b. So ways these properties can be are naturally represented by access to arbitrary pairs. But we have seen the first-degree relevant picture driven by a notion of property inclusion, where one property is "followed" by another when it is *included* as a constituent of the other. And access to individual worlds may seem sufficient to represent property inclusions. However this is to leave out an important part of the picture. Not every entailment or inclusion property is instantiated. Where entailment properties are constituents of closed properties, and properties come in arbitrary combinations, the range of entailment properties is broader than the range of entailments. Correspondingly, the range of entailment properties is modeled only over a range of property pairs broader than the range of actual inclusions. Thus we need the power to represent *arbitrary* closed properties as standing in this relation — very much as in the necessary t-following and unambiguous preference cases. So we let a world w represent that x is a constituent of a closed y when it has access to an arbitrary pair $\langle x, y\rangle$.

3.2 Basic System: *4A*

For this, consider a language like the one for *FB*, but without constraints on formation rules for $\rightarrow$. We treat $\supset$ as an abbreviation, defined in terms of $\neg$ and $\vee$ in the usual way. An interpretation is $\langle W, R, \overline{R}, v\rangle$ where W is a set of worlds, $R, \overline{R} \subseteq W^3$ are relevant access relations for truth and non-falsity respectively; and v is a valuation which assigns to each parameter some subset of $\{1, 0\}$ at each $w \in W$. Definition H4 carries over semantic conditions for parameters, $\neg$, $\wedge$, and $\vee$ from HFB. Then,

H4 $(\rightarrow)$ $h_w(/P \rightarrow Q/) = 1$ iff there are no $x, y \in W$ such that $w/R/xy$ and either $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$.

Where the members of Γ and P are without overlines, $h_w(\Gamma) = 1$ iff $h_w(A) = 1$ for each $A \in \Gamma$; and,

V4A $\Gamma \models_{4A} P$ iff there is no *4A* interpretation $\langle W, R, \overline{R}, v\rangle$ and $w \in W$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

The move to ternary access allows truth and falsity for arbitrary conditionals. Separating R and $\overline{R}$ allows models of closed properties where truth and (non)falsity for conditionals are independent. So we model closed properties with neither BEING P $\rightarrow$ Q nor BEING $\neg$(P $\rightarrow$ Q) as constituents, and ones with both.

Observe that H4($\rightarrow$) reflects the move to world-pairs, and the picture of access to a pair $\langle x, y\rangle$ as representing the inclusion of x in y. On our general picture for entailment, if BEING P entails BEING Q, (i) BEING P is included in BEING Q — so that if P holds at x, Q holds at y, and (ii) with negations as compliments, BEING $\neg$Q is included in BEING $\neg$P — so that if $\neg Q$ holds at x, $\neg P$ holds at y — or alternatively if $\neg P$ does not hold at y (if $\overline{P}$ holds at y) then $\neg Q$ does not hold at x ($\overline{Q}$ holds at x). Thus the inclusion picture leads to the double condition as above.

For a derivation system, allow expressions of the sort, $s.t.u$ and $\overline{s.t.u}$. Intuitively, $/s.t.u/$ just in case $s/R/tu$. Rules R, $\neg$I, $\neg$E, $\wedge$I, $\wedge$E, $\vee$I, and $\vee$E (along with $\supset$I and $\supset$E) carry over from *FB*. Then,

$$\rightarrow\!\mathbf{E}\quad \begin{array}{|l} /s.t.u/ \\ /P \rightarrow Q/_s \\ P_t \\ \\ Q_u \end{array} \qquad \begin{array}{|l} /s.t.u/ \\ /P \rightarrow Q/_s \\ \overline{P}_u \\ \\ \overline{Q}_t \end{array} \qquad\qquad \rightarrow\!\mathbf{I}\quad \begin{array}{|l} \begin{array}{|l} /s.t.u/ \\ \underline{P_t} \\ \\ Q_u \end{array} \\ /P \rightarrow Q/_s \end{array} \qquad \begin{array}{|l} \begin{array}{|l} /s.t.u/ \\ \underline{\overline{P}_u} \\ \\ \overline{Q}_t \end{array} \\ /P \rightarrow Q/_s \end{array}$$

where t and u do not appear in any undischarged premise or assumption

As before, any subscript is 0 or introduced as t or u in an assumption for $\rightarrow$I. Where Γ is a set of unsubscripted expressions without overlines, let

Γ_0 be those same expressions, each with subscript 0. Then $\Gamma \vdash_{4A} P$ iff there is a *4A* derivation of P_0 from the members of Γ_0. Here is an example beyond what can be established or even represented in *FB*. $A \to (B \to C)$, $A \to (D \to \neg B) \vdash_{4A} A \to [B \to (C \wedge \neg D)]$.

1	$(A \to (B \to C))_0$	prem
2	$(A \to (D \to \neg B))_0$	prem
3	0.1.2	A ($\to$I)
4	A_1	
5	$(B \to C)_2$	3,1,4 $\to$E
6	$(D \to \neg B)_2$	3,2,4 $\to$E
7	2.3.4	A ($\to$I)
8	B_3	
9	C_4	7,5,8 $\to$E
10	$\overline{D}_4$	A ($\neg$I)
11	$\overline{\neg B}_3$	7,6,10 $\to$E
12	B_3	8 R
13	$\neg D_4$	10-12 $\neg$I
14	$(C \wedge \neg D)_4$	9,13 $\wedge$I
15	$[B \to (C \wedge \neg D)]_2$	7-14 $\to$I
16	$(A \to [B \to (C \wedge \neg D)])_0$	3-15 $\to$I

The derivation system is sound and complete, $\Gamma \vdash_{4A} P$ iff $\Gamma \models_{4A} P$ (Roy, draftB, §9). And we recover the relevant system *DW* as *FA* recovers FDE, $P \vdash_{4A} Q$ iff $\vdash_{DW} P \to Q$ [A3].

As one might expect, nothing prevents imposing constraints on R and $\overline{R}$ to reach certain other relevant systems. Thus corresponding to standard constraints from the star semantics, consider,

D3/4 If $a/R/bx$ and $xRcd$ then there is a y such that $bRcy$ and $a/R/yd$, and a z such that $bRzd$ and $a/R/cz$. And if $a/R/xb$ and $x\overline{R}cd$ then there is a y such that $b\overline{R}cy$ and $a/R/yd$, and a z such that $b\overline{R}zd$ and $a/R/cz$.

D5 If $a/R/bc$ then there is a y such that $a/R/by$ and $yRbc$ and a z such that $a/R/zc$ and $z\overline{R}bc$.

And, given a family of *inclusions* such that if $a \leq b$ then if P holds at a, P holds at b; if $a \leq^* b$ then if P holds at a, $\overline{P}$ holds at b; and if $a \leq^\sharp b$ then if $\overline{P}$ holds at a, P holds at b,

D6 If $aRbc$ then for some $y \geq a$, $bRyc$, and for some $z \geq^* a$, $c\overline{R}bz$. And if $a\overline{R}bc$ then for some $y \geq^\sharp a$, $bRyc$, and for some $z \leq a$, $c\overline{R}bz$.[12]

Such principles may be incorporated into derivations, and enable contact with other standard relevant systems.

There may be motivations from the inclusion picture for something like D3/4.[13] I see no basis for D5 or D6. But that is not terribly important here. It is, no doubt, a technical advantage to accommodate a wide range of formal systems from a single semantic framework. However, insofar as we are after an account of entailment in the sense of LV, that we can impose some constraints to achieve other relevant systems is surely not sufficient

[12] These numbers parallel say Restall (1993) and Routley (1984). This last condition diverges from proposals considered in Routley (1984). It is however relatively natural given the parallel condition required for the simplified version of the star semantics as discussed in Restall and Roy (2009). For discussion see Roy (draftA).

[13] Consider the first clause. Suppose $a/R/bx$ and $xRcd$; then according to world a, b is included in x; so according to a any entailment property in b is included in x; so if x has access to some $\langle c, d\rangle$ that counterexample some conditional, then a must "think" b has access to worlds with this effect — worlds that counterexample all the same conditionals. Now consider the constraint,

$$\begin{tikzcd}
 & \text{exists } y,\ a/R/yd \arrow[dr, dash] & \\
a/R/bx,\ xRcd \arrow[ur] \arrow[dr] & & bRcy \quad bRzd \quad (bRcd) \\
 & \text{exists } z,\ a/R/cz \arrow[ur, dash] &
\end{tikzcd}$$

Suppose $a/R/bx$ and $xRcd$. We are not given $bRcd$. However, there is a y that a thinks is included in d such that $bRcy$, and a z such that a thinks c is included in it where $bRzd$. But if y is included in d and $bRcy$, then b has access at least to counterexamples for all the same conditionals as x from $xRcd$; and if c is included in z and $bRzd$, then b has access at least to counterexamples for all the same conditionals as x from $xRcd$. In either case, there are worlds to which b bears R and, according to a, give b access to the very counterexamples x has by $xRcd$. So these are sufficient for a to sustain its claim that b is included in x. Similar reasoning applies to the other clause.

Another natural consequence from this reasoning might seem to be $bRzy$. This verifies $[(A \to B) \wedge (C \to D)] \to [(B \to C) \to (A \to D)]$ in standard systems.

to say we should. Clearly the situation for constraints on a ternary R is complex. Perhaps there is no semantic motivation for stronger constraints. And this may be reason to reject them as part of an account of entailment.

A few comments: First, in his presentation of related systems (as in the next section), Routley represents the different relevant access relations R and $\overline{R}$ as a sort of cost relative to the star semantics, which requires only one. Perhaps there is a sort of cost, but I do not think it is substantive. In the star semantics, apart from special constraints on access, worlds accessible to w are independent of worlds accessible to w^*. So worlds relevant to the truth of $A \rightarrow B$ at w are independent of worlds relevant to its non-falsity. So the effect is as with R and $\overline{R}$. On either account, independent access makes possible models for arbitrary entailment properties.

Similarly, one might think the disjunctive condition for H4($\rightarrow$) is a cost relative to the star semantics, which makes the truth of $A \rightarrow B$ at w depend on just whether there are $x, y \in W$ such that $wRxy$ where A is true at x but B not at y. However relevant systems as strong as DW require that if $wRxy$ then wRy^*x^*. But then, where P's truth at w^* corresponds to its non-falsity at w, the truth of a conditional $A \rightarrow B$ at w generally requires that there is no $\langle x, y\rangle$ to which w has access such that A is true at x but B is not at y, or A is not false at y but B is false at x. So the effect is the same. We have seen the double condition from the first-degree account of entailment and from the picture of properties. Perhaps, then, the four-valued approach, for all its relative inelegance, is useful insofar as it brings to the surface how different conditions work.

Finally, one might think star worlds are themselves a cost relative to the four-valued approach. I think they are a cost — at least insofar as one is interested in seeing connection to domains of closed properties. But technically, for systems considered here, symmetries are such that star worlds "do no harm." Roughly, the point is this: Say $A \rightarrow B$ holds at w. Then on our account there are no x, y such that $wRxy$ where A holds at x but B does not hold at y, or $\overline{A}$ holds at y but $\overline{B}$ does not hold at x. But then the conditional continues to hold if there are worlds x' and y' such that truths and non-falsities are exchanged relative to x and y and $wRy'x'$. For then the "truth" condition on $\langle x, y\rangle$ corresponds to the non-falsity one on $\langle y', x'\rangle$, and the non-falsity condition on $\langle x, y\rangle$ to the truth condition on $\langle y', x'\rangle$. So results need not change when models are such that any world has a mate with truths and non-falsities exchanged. One might let pairs $\langle w, w^*\rangle$ model closed properties. But this is at least obscure relative to our picture on which worlds correspond to them directly.

On either approach, then, it is possible adopt our basic picture: There is a domain $\mathcal{D}$ of closed properties, now with entailment properties as constituents. P logically entails Q just in case there is no BEING W $\in \mathcal{D}$ such that BEING P is a constituent of BEING W but BEING Q is not. These conditions are modeled by the semantics for *4A* (or perhaps some *4Ax* with appropriate constraints on R and $\overline{R}$). And the derivation system is adequate to the semantics. So again we seem to have all the elements to represent the approach as reflecting an *account* of entailment.

3.3 Results Revised: *4B*

As before, what we have is not the same as standard relevant systems. *DW* for example, is developed as an axiom system with derivations and theorems in the usual way. But *4A* has no theorems. Rather, $P \vdash_{4A} Q$ iff $\vdash_{DW} P \rightarrow Q$. But we can convert our valid arguments into theorems. Again, this comes to a sort of transform, useful insofar as it puts us in a position to draw further connections and consequences.

For a complete description of the resultant system, see Appendix B. Basically, an interpretation is $\langle W, N, \overline{N}, R, \overline{R}, v\rangle$ where $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively. As a constraint on interpretations, we require a *normality condition*,

NC For any $w \in /N/$, $w/R/xy$ iff $x = y$.

And there may be constraints on R and $\overline{R}$ as before. Then where the members of Γ and P are without overlines, and x is empty or indicates some combination of optional constraints,

V4Bx $\Gamma \models_{4Bx} P$ iff there is no *4Bx* interpretation $\langle W, N, \overline{N}, R, \overline{R}, v\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

Given NC, the condition for $P \to Q$ at a normal world "looks" like the condition for validity in $\mathit{4Ax}$. At a normal world, $P \to Q$ just in case no individual world is such that $/\!\!/P/\!\!/$ holds but $/\!\!/Q/\!\!/$ does not.

In addition to rules from before, to accommodate normality derivations require,

NE	$/n/[a]$	$/n/[a]$	$\simeq$**I**		$\simeq$**E**	$s \simeq t$
	$s \simeq t$	$/a.s.t/$				$\mathcal{P}(s)$
	$/a.s.t/$	$s \simeq t$		$s \simeq s$		$\mathcal{P}(t)$

Then, corresponding to the above example for $\mathit{4A}$ we have, $\vdash_{4B} ([A \to (B \to C)] \wedge [A \to (D \to \neg B)]) \to (A \to [B \to (C \wedge \neg D)])$.

1	$0.1.2$	A ($\to$I)
2	$([A \to (B \to C)] \wedge [A \to (D \to \neg B)])_1$	
3	$n[0]$	NI
4	$1 \simeq 2$	1,3 NE
5	$([A \to (B \to C)] \wedge [A \to (D \to \neg B)])_2$	2,4 $\simeq$E
6	$[A \to (B \to C)]_2$	5 $\wedge$E
7	$[A \to (D \to \neg B)]_2$	5 $\wedge$E
8	$2.3.4$	A ($\to$I)
9	A_3	
10	$(B \to C)_4$	8,6,9 $\to$E
11	$(D \to \neg B)_4$	8,7,9 $\to$E
12	$4.5.6$	A ($\to$I)
13	B_5	
14	C_6	12,10,13 $\to$E
15	$\overline{D}_6$	A ($\neg$I)
16	$\overline{\neg B}_5$	12,11,15 $\to$E
17	B_5	13 R
18	$\neg D_6$	15-17 $\neg$I
19	$(C \wedge \neg D)_6$	14,18 $\wedge$I
20	$[B \to (C \wedge \neg D)]_4$	12-19 $\to$I
21	$(A \to [B \to (C \wedge \neg D)])_2$	8-20 $\to$I
22	$[([A \to (B \to C)] \wedge [A \to (D \to \neg B)]) \to (A \to [B \to (C \wedge \neg D)])]_0$	1-21 $\to$I

Derivations are sound and complete, $\Gamma \vdash_{4Bx} P$ iff $\Gamma \models_{4Bx} P$ (Roy, draftB, §9). Where Sx is a standard relevant system from DW, TW, RW and R with constraints parallel to D3 - D6, $\Gamma \models_{4Bx} P$ iff $\Gamma \models_{Sx} P$ (Roy, draftA). Where x includes CL, and the derivation system (C), we recover all of classical logic, so that for classical forms, $\Gamma \vdash_{CL} P$ iff $\Gamma \vdash_{4Bx} P$. In addition, nothing prevents expanding the language to include $\Box$ (and $\Diamond$) with interpretations, $\langle W, M, N, \overline{N}, R, \overline{R}, v\rangle$ where M is a two-place modal access relation — again possibly restricted in the usual way. Then, developed as before, we recover also all of ordinary modal logic.[14]

Again, the move from *4A* to *4B* effects a sort of conversion on results. Entailments from *4A* appear as theorems in *4B* [A3]. So we retain the sort of motivations for theorems that have application to *4A*. Again, though, the notion of conversion only takes us so far. So it leaves open what to make of normal worlds and validity more generally in the *4Bx* systems. And the *4B* normality constraint NC seems to require of conditionals classical behavior for truth at normal worlds, but not for falsity, and not for anything else. This is a peculiar asymmetry which recapitulates what we saw from *FB* and seems unmotivated from our perspective. The asymmetry is nowhere to be found in *FA* or *4A*. And, as for *FBc*, it is removed in a *4Bc*, which accepts (CL) and so identifies N and $\overline{N}$, and builds in also a clarified role for normal worlds as world-like. Again in *4Bc*, as in in *FBc*, '$\rightarrow$' appears just as an additional operator strengthening and extending a standard tool set as does, say, '$\Box$' in the ordinary case. Perhaps, then, the more natural paraconsistent systems are *FA* and *4A*, and natural "converted" systems are something like *FBc* and *4Bc*, none of which involve the asymmetry.[15]

[14]This time we acquire also full-fledged *relevant* modal logics with characteristic principles as T: $\Box A \rightarrow A$; B: $A \rightarrow \Box\Diamond A$ and 4: $\Box A \rightarrow \Box\Box A$ with the standard constraints on access. Without constraint, $A \rightarrow B \models_{4Bx} \Box A \rightarrow \Box B$; $(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ comes with an additional constraint, $(\kappa\rho)$ if $a/R/bx$ and xMc then there is a y such that bMy and $a/R/yc$, and if $a/R/xb$ and xMc, there is a y such that bMy and $a/R/cy$; the K principle $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ then comes together with reflexivity. This constraint has a motivation like that for D3/4 (see note 13). It is possible to obtain the K principle independently, but the required constraint with conditions like, (κ) if $a/R/bx$ and xMc, then there are y, z such that aMy, bMz and $y/R/zc$, is not so easy to motivate. For discussion see Fuhrmann (1990).

[15]The reason for dividing N and $\overline{N}$, both here and for *FB*, is to make contact with

4 Conclusion

I do not say there is a direct path from the initial picture of logical entailment to relevant systems like DW,TW, *4A* or *4Bc*. One might end up with related systems, or ones quite different. I do say relevant systems like ones we have considered make *sense* as based on the theory of entailment. The relevant approach grounds entailment in property inclusion, and so offers an account of intuitions with which we began in terms of this notion; inclusion or something like it is already required for classical validity and to this extent available; and the semantics is matched to objects of philosophical analysis so as to lay claim to a legitimate and fundamental notion of entailment.[16] The relevant systems offer a way to see classical logic as arising from the account of entailment, subject to constraints that matter. Thus the relevant derivations and semantics appear as more than merely *ad hoc* systems to satisfy certain syntactical intuitions. Rather, whatever its relation to syntactical intuitions, the relevant approach seems grounded in a sensible semantic picture. Insofar as the semantic picture satisfies also syntactical intuitions, the result may be a pleasing reflective equilibrium.

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standard relevant systems. Having made contact, a consequence may be grounds for criticism — or, at least, apart from the point about conversion, I have offered no account of normal worlds in the divided case.

[16]There might be an account on which relevant semantics models *theories* or *information* that is isomorphic to the one offered here. In this case, I have no decisive objection — and perhaps reason to applaud. Still, there may be considerations in favor of the present approach: (i) Validity is defined classically on a universe of worlds. The shift to properties is continuous with that account — both in the sense that it accommodates classical validity, and that it matters similarly for reasoning about the world. (ii) Information is not intrinsically constrained so that when it includes $p \vee q$ then it includes p or q. So information has to be constrained in ways that may seem *ad hoc*. The parallel constraints fall naturally out of the account based on closed properties. (iii) There might very well be an informational notion of *containment* parallel to property inclusion. However (crucial for making sense of relevant semantics) if, as above, the relevant access relation is given its account in terms of "entailments" already present in the information, the account will not illuminate entailment for those who do not already understand it. The "metaphysical" picture makes progress toward illumination.

Appendix A

A1 $P \vdash_{FA} Q$ iff $\vdash_{FB} P \to Q$. Given that the systems are sound and complete: Suppose $P \nvDash_{FA} Q$; then there is an interpretation v such that $1 \in v(P)$ but $1 \notin v(Q)$; but then an *FB* interpretation, $\langle W, N, \overline{N}, v' \rangle$ with some $w \in W$ such that $v'_w(p) = v(p)$ has $h_w(P) = 1$ and $h_w(Q) = 0$; so $\nvDash_{FB} P \to Q$. Suppose $\nvDash_{FB} P \to Q$; then there is an interpretation $\langle W, N, \overline{N}, v' \rangle$ with some $w \in W$ such that $h_w(P) = 1$ and $h_w(Q) = 0$ or $h_w(\overline{P}) = 1$ and $h_w(\overline{Q}) = 0$; in the first case, an *FA* interpretation v with $v(p) = v'_w(p)$ has $1 \in v(P)$ but $1 \notin v(Q)$; in the second case, an interpretation v^* constructed as in note 5 has $1 \in v^*(P)$ but $1 \notin v^*(Q)$; in either case, $P \nvDash_{FA} Q$.

A2 In the language for *FB*, $\Gamma \vdash_{FB} P$ iff $\Gamma \vdash_{Sx} P$. Consider the standard simplified semantics for relevant logic as in Restall (1993); Restall and Roy (2009); Priest (2001), but numbers for conditions are different in Priest. Given that the systems are sound and complete: (i) Suppose $\Gamma \nvDash_{Sx} P$; then there is a relevant interpretation $\langle \mathsf{W}, \mathsf{g}, \leq, \mathsf{R}, \star, \mathsf{v} \rangle$ such that $\mathsf{g}(\Gamma) = 1$ but $\mathsf{g}(P) = 0$. Corresponding to $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \leq \mathsf{v} \rangle$ consider $\langle W, N, \overline{N}, v \rangle$ with $w \in W$ corresponding to each $\mathsf{w} \in \mathsf{W}$; set $N = \{g\}$, and $\overline{N} = \phi$; let $1 \in v_w(p)$ iff $\mathsf{v_w}(p) = 1$ and $0 \notin v_w(p)$ iff $\mathsf{v_{w^\star}}(p) = 1$; then for $w \notin N$ set $h_w(P \to Q) = \mathsf{v_w}(P \to Q)$ and for $w \notin \overline{N}$, $h_w(\overline{P \to Q}) = \mathsf{v_{w^\star}}(P \to Q)$. Then by a straightforward induction, $h_g(\Gamma) = 1$ but $h_g(P) = 0$; so $\Gamma \nvDash_{FB} P$. (ii) Suppose $\Gamma \nvDash_{FB} P$; then there is $\langle W, N, \overline{N}, v \rangle$ with $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$; first generate a parallel interpretation with $N = \{w\}$ and $\overline{N} = \phi$, setting arbitrary assignments so that all the same formulas hold at all the same worlds. Then consider $\langle \mathsf{W}, \mathsf{g}, \mathsf{R}, \star, \leq, \mathsf{v} \rangle$ where for each $a \in W$ there are $\mathsf{a}, \mathsf{a}^+ \in \mathsf{W}$ with $\mathsf{g} = \mathsf{w}$; $\star$ the set of all pairs $\langle \mathsf{a}, \mathsf{a}^+ \rangle$, $\langle \mathsf{a}^+, \mathsf{a} \rangle$; $\mathsf{v_a}(p) = 1$ iff $1 \in v_a(p)$ and $\mathsf{v_{a^+}}(p) = 1$ iff $0 \notin v_a(p)$. Now, simply picking access relations that will satisfy the relevant conditions: (a) To capture D20 with D1 - D5 and so any of B, DW and TW, set $\mathsf{w} \geq \mathsf{w}$ and $\mathsf{R} = \{\langle \mathsf{g}, \mathsf{x}, \mathsf{x} \rangle, \langle \mathsf{x}, \mathsf{x}, \mathsf{x} \rangle \mid \mathsf{x} \in \mathsf{W}\}$; (b) To capture D20, D3, D4, and D6 and so RW set $\mathsf{w} \geq \mathsf{w}$ and $\mathsf{R} = \{\langle \mathsf{g}, \mathsf{x}, \mathsf{x} \rangle, \langle \mathsf{x}, \mathsf{g}, \mathsf{x} \rangle, \langle \mathsf{x}, \mathsf{x}^\star, \mathsf{g}^\star \rangle \mid \mathsf{x} \in \mathsf{W}\}$. Then, in any case, for $w \notin N$ reset $h_w(P \to Q) = \mathsf{v_w}(\mathsf{P} \to \mathsf{Q})$. This is an *Sx* interpretation such that $\mathsf{v_g}(\Gamma) = 1$ but $\mathsf{v_g}(P) = 0$; so $\Gamma \nvDash_{Sx} P$.

A3 $P \vdash_{4Ax} Q$ iff $\vdash_{4Bx} P \to Q$. Given that the systems are sound and complete: (i) Suppose $\nvDash_{4Bx} P \to Q$; then there is a *4Bx* interpretation $\langle W, N, \overline{N}, R, \overline{R}, v \rangle$, and $w \in N$ such that $h_w(P \to Q) = 0$; so by H4B($\to$), there are $x, y \in W$ such that $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$; since $w \in N$, by NC, $x = y$, so there is an $x \in W$ such that either $h_x(P) = 1$ but $h_x(Q) = 0$, or $h_x(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$. By a lemma from the demonstration of soundness, there is a corresponding *4Bx* interpretation $\langle \mathsf{W}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \mathsf{v} \rangle$

with w and w^* corresponding to each w such that $\mathsf{h}_{\mathsf{w}}(/A/) = 1$ iff $h_w(/A/) = 1$ and $\mathsf{h}_{\mathsf{w}^*}(/A/) = 1$ iff $h_w(\backslash A\backslash) = 1$. So $\mathsf{h}_{\mathsf{x}}(P) = 1$ and $\mathsf{h}_{\mathsf{x}}(Q) = 0$, or $\mathsf{h}_{\mathsf{x}^*}(P) = 1$ and $\mathsf{h}_{\mathsf{x}^*}(Q) = 0$. One way or the other, there is an $\mathsf{a} \in \mathsf{W}$ such that $\mathsf{h}_{\mathsf{a}}(P) = 1$ and $\mathsf{h}_{\mathsf{a}}(Q) = 0$. But then, by a simple induction, a parallel *4Ax* interpretation with w, R, $\overline{R}$, and v all the same has $h_a(P) = 1$ but $h_a(Q) = 0$. So $P \not\models_{4Ax} Q$. (ii) Suppose $P \not\models_{4Ax} Q$; then there is a *4Ax* interpretation $\langle W, R, \overline{R}, v\rangle$ and $w \in W$ such that $h_w(P) = 1$ but $h_w(Q) = 0$. Consider $\langle \mathsf{W}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \mathsf{v}\rangle$ where $\mathsf{W} = W \cup \{n\}$; $\mathsf{N} = \{n\}$; $\overline{\mathsf{N}} = \phi$; $\mathsf{R} = R \cup \{\langle n, w, w\rangle \mid w \in \mathsf{W}\}$; $\overline{\mathsf{R}} = \overline{R} \cup \{\langle n, w, w\rangle \mid w \in \mathsf{W}\}$; and v is extended to paramaters at n in arbitrary ways. If $\langle W, R, \overline{R}, v\rangle$ is a *4Ax* interpretation, then $\langle \mathsf{W}, \mathsf{N}, \overline{\mathsf{N}}, \mathsf{R}, \overline{\mathsf{R}}, \mathsf{v}\rangle$ is a *4Bx* interpretation. But then, $n\mathsf{R}ww$ and, by a simple induction, $\mathsf{h}_w(P) = 1$ but $\mathsf{h}_w(Q) = 0$; so by HFB($\rightarrow$), $\mathsf{h}_n(P \rightarrow Q) = 0$; so $\not\models_{4Bx} P \rightarrow Q$. This last result extends only to logics with conditions D1 - D5, and not all the way to R, as consideration of an expression like $[(p \rightarrow p) \rightarrow q] \rightarrow q$ will show.

Appendix B

Complications for systems up to R are treated in Roy (draftB). Here I collect elements of the main system from this paper.

B1 The VOCABULARY consists of propositional parameters $p_0, p_1 \ldots$ with the operators $\neg$, $\wedge$, $\vee$, $\Box$ and $\rightarrow$. Each propositional parameter is a FORMULA; if A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $\Box A$ and $(A \rightarrow B)$. $A \supset B$ abbreviates $\neg A \vee B$, $\Diamond A$ abbreviates $\neg\Box\neg A$. If A is a formula so formed, so is $\overline{A}$.

Let $/A/$ and $\backslash A\backslash$ represent either A or $\overline{A}$ where what is represented is constant in a given context, but $/A/$ and $\backslash A\backslash$ are opposite. And similarly for other expressions with overlines as below.

B2 An INTERPRETATION is $\langle W, M, N, \overline{N}, R, \overline{R}, v\rangle$ where W is a set of worlds; $M \subseteq W$ is a modal access relation; $N, \overline{N} \subseteq W$ are normal worlds for truth and non-falsity respectively; $R, \overline{R} \subseteq W^3$ are access relations for truth and non-falsity respectively; and v is a valuation which assigns to each $/p/$ some subset of $\{1, 0\}$ at each $w \in W$. Interpretations are subject to,

NC For any $w \in /N/$, $w/R/xy$ iff $x = y$

MC If $w \in /N/$ and wMx, then $x \in /N/$

and optionally,

D3/4 If $a/R/bx$ and $xRcd$ then there is a y such that $bRcy$ and $a/R/yd$, and a z such that $bRzd$ and $a/R/cz$. And if $a/R/xb$ and $x\overline{R}cd$ then there is a y such that $b\overline{R}cy$ and $a/R/yd$, and a z such that $b\overline{R}zd$ and $a/R/cz$

CL (i) $w \in N$ iff $w \in \overline{N}$

(ii) for any $w \in N$, $h_w(\overline{p}) = h_w(p)$

MS $\kappa\rho$ if $a/R/bx$ and xMc then there is a y such that bMy and $a/R/yc$, and if $a/R/xb$ and xMc, there is a y such that bMy and $a/R/cy$

ρ Reflexivity: for all x, xMx

σ Symmetry: for all x, y if xMy then yMx

τ Transitivity: for all x, y, z if xMy and yMz then xMz

B3 For complex expressions,

(B) $h_w(p) = 1$ iff $1 \in v_w(p)$; $h_w(\overline{p}) = 1$; iff $0 \notin v_w(p)$

($\neg$) $h_w(/\neg P/) = 1$ iff $h_w(\backslash P\backslash) = 0$

($\wedge$) $h_w(/P \wedge Q/) = 1$ iff $h_w(/P/) = 1$ and $h_w(/Q/) = 1$

($\vee$) $h_w(/P \vee Q/) = 1$ iff $h_w(/P/) = 1$ or $h_w(/Q/) = 1$

($\rightarrow$) $h_w(/P \rightarrow Q/) = 1$ iff there are no $x, y \in W$ such that $w/R/xy$ and $h_x(P) = 1$ but $h_y(Q) = 0$, or $h_y(\overline{P}) = 1$ but $h_x(\overline{Q}) = 0$

($\Box$) $h_w(/\Box P/) = 1$ iff there is no $x \in W$ such that wMx and $h_x(/P/) = 0$

For a set Γ of formulas, $h_w(\Gamma) = 1$ iff $h_w(/P/) = 1$ for each $/P/ \in \Gamma$; then,

V4Bx $\Gamma \models_{4Bx} P$ iff there is no 4Bx interpretation $\langle W, M, N, \overline{N}, R, \overline{R}, v\rangle$ and $w \in N$ such that $h_w(\Gamma) = 1$ but $h_w(P) = 0$.

B4 For all the rules of 4B, allow overlines, subscripts, and expressions of the sort $s.t$, $/r.s.t/$, $/n/[s]$, and $s \simeq t$.

$$\begin{array}{r|l} \textbf{R} & /P/_s \\ & \\ & /P/_s \end{array} \qquad \begin{array}{r|l} \neg\textbf{I} & \begin{array}{|l} \underline{/P/_s} \\ \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash\neg P\backslash_s \end{array} \qquad \begin{array}{r|l} \neg\textbf{E} & \begin{array}{|l} \underline{/\neg P/_s} \\ \\ /\!/Q/\!/_t \\ \backslash\!\backslash\neg Q\backslash\!\backslash_t \end{array} \\ & \backslash P\backslash_s \end{array}$$

$$\begin{array}{r|l} \wedge\textbf{I} & /P/_s \\ & /Q/_s \\ & \\ & /P \wedge Q/_s \end{array} \qquad \begin{array}{r|l} \wedge\textbf{E} & /P \wedge Q/_s \\ & \\ & /P/_s \end{array} \qquad \begin{array}{r|l} \wedge\textbf{E} & /P \wedge Q/_s \\ & \\ & /Q/_s \end{array}$$

$$\vee\mathbf{I}\;\begin{array}{|l} /P/_s \\ \\ /P \vee Q/_s \end{array} \qquad \vee\mathbf{I}\;\begin{array}{|l} /P/_s \\ \\ /Q \vee P/_s \end{array} \qquad \vee\mathbf{E}\;\begin{array}{|l} /P \vee Q/_s \\ \begin{array}{|l} \underline{/P/_s} \\ \\ /\!/R/\!/_t \end{array} \\ \begin{array}{|l} \underline{/Q/_s} \\ \\ /\!/R/\!/_t \end{array} \\ /\!/R/\!/_t \end{array}$$

$$\supset\mathbf{I}\;\begin{array}{|l} \begin{array}{|l} \underline{/P/_s} \\ \\ \backslash Q\backslash_s \end{array} \\ \backslash P \supset Q\backslash_s \end{array} \qquad \supset\mathbf{E}\;\begin{array}{|l} \backslash P \supset Q\backslash_s \\ /P/_s \\ \\ \backslash Q\backslash_s \end{array}$$

$$\rightarrow\mathbf{E}\;\begin{array}{|l} /s.t.u/ \\ /P \rightarrow Q/_s \\ P_t \\ \\ Q_u \end{array} \quad \begin{array}{|l} /s.t.u/ \\ /P \rightarrow Q/_s \\ \overline{P}_u \\ \\ \overline{Q}_t \end{array} \qquad \rightarrow\mathbf{I}\;\begin{array}{|l} \begin{array}{|l} /s.t.u/ \\ \underline{P_t} \\ \\ Q_u \end{array} \\ /P \rightarrow Q/_s \end{array} \quad \begin{array}{|l} \begin{array}{|l} /s.t.u/ \\ \underline{\overline{P}_u} \\ \\ \overline{Q}_t \end{array} \\ /P \rightarrow Q/_s \end{array} \qquad \mathbf{C}\;\begin{array}{|l} /n/[s] \\ /\!/P/\!/_s \\ \\ \backslash\!\backslash P\backslash\!\backslash_s \end{array} \quad \begin{array}{|l} /n/[s] \\ \\ \\ \backslash n\backslash[s] \end{array}$$

where t and u do not appear in any undischarged premise or assumption

$$\mathbf{NI}\;\begin{array}{|l} \\ \\ n[0] \end{array} \qquad \mathbf{NE}\;\begin{array}{|l} /n/[a] \\ s \simeq t \\ \\ /a.s.t/ \end{array} \quad \begin{array}{|l} /n/[a] \\ /a.s.t/ \\ \\ s \simeq t \end{array} \qquad \simeq\mathbf{I}\;\begin{array}{|l} \\ \\ s \simeq s \end{array} \qquad \simeq\mathbf{E}\;\begin{array}{|l} s \simeq t \\ \mathcal{P}(s) \\ \\ \mathcal{P}(t) \end{array}$$

$$\Box\mathbf{I}\;\begin{array}{|l} \begin{array}{|l} \underline{s.t} \\ \\ /P/_t \end{array} \\ /\Box P/_s \end{array} \qquad \Box\mathbf{E}\;\begin{array}{|l} /\Box P/_s \\ s.t \\ \\ /P/_t \end{array} \qquad \mathbf{MC}\;\begin{array}{|l} /n/[s] \\ s.t \\ \\ /n/[t] \end{array} \qquad \mathbf{AM}\rho\;\begin{array}{|l} \\ \\ \\ s.s \end{array} \qquad \mathbf{AM}\sigma\;\begin{array}{|l} s.t \\ \\ \\ t.s \end{array} \qquad \mathbf{AM}\tau\;\begin{array}{|l} s.t \\ t.u \\ \\ s.u \end{array}$$

where t does not appear in any undischarged premise or assumption

$$\mathbf{AM}^{\kappa}_{\rho}\;\begin{array}{|l} /a.b.x/ \\ x.c \\ \begin{array}{|l} b.y \\ \underline{/a.y.c/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \quad \begin{array}{|l} /a.x.b/ \\ x.c \\ \begin{array}{|l} b.y \\ \underline{/a.c.y/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \quad \mathbf{AM}^{3}_{4}\;\begin{array}{|l} /a.b.x/ \\ x.c.d \\ \begin{array}{|l} b.c.y \\ \underline{/a.y.d/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \quad \begin{array}{|l} /a.b.x/ \\ x.c.d \\ \begin{array}{|l} b.y.d \\ \underline{/a.c.y/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \quad \begin{array}{|l} /a.x.b/ \\ \overline{x.c.d} \\ \begin{array}{|l} \overline{b.c.y} \\ \underline{/a.y.d/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array} \quad \begin{array}{|l} /a.x.b/ \\ \overline{x.c.d} \\ \begin{array}{|l} \overline{b.y.d} \\ \underline{/a.c.y/} \\ \\ /\!/P/\!/_w \end{array} \\ /\!/P/\!/_w \end{array}$$

where y does not appear in any undischarged premise or assumption and is not w

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